University of Waterloo – CO 353 Practice Final (Winter 2018)

Course:	CO 353 – Computational discrete optimization
Date and time of exam:	Tuesday April 24, 2018 – 12:30 PM to 3:00 PM
Duration of exam:	2h30
Location:	PAC Upper 9
Exam type:	Faculty of Math approved calculators can be used
	Closed book – no other materials allowed

Your answers must be stated and justified in a clear and logical form, and you must show all of your steps in order to receive full marks. You may use any result from class without proof, unless you are being asked to prove this result. You will be graded not only on correctness, but also on clarity of exposition. No collaboration is allowed.

Question 1a Consider the following algorithm, that takes as an input a matrix $A \in \mathbb{Z}^{m \times n}$, with $A_{\max} := \max_{i,j} |A_{ij}|$.

```
for  i = 1,...n
|    for  j = 1,...n
|    |    v := 0
|    |    for  k = 1,...m
|    |    |    v := v + A_ik A_jk
|    |    G_ij := v
return  G
```

1. Give the encoding length L of the input, in big-O notation.
2. Using the arithmetic model, give the algorithmic complexity of the algorithm, in big-O notation.
3. Is the complexity polynomial in the input length L? If it is, give the polynomial in big-O notation. Otherwise, explain why it is not polynomial.

Question 1b Consider the following algorithm, that takes a positive number $a \in \mathbb{Z}$ in input.

```
f := {}
r := a
i := 2
while  i ≤ r
|    if  (r/i) ∈ Z
|    |    f := f ∪ {i}
|    |    r := r/i
|    else
|    |    i := i + 1
return  f
```

1. Give the encoding length L of the input, in big-O notation.
2. Using the arithmetic model, give the algorithmic complexity of the algorithm, in big-O notation.
3. Is the complexity polynomial in the input length L? If it is, give the polynomial in big-O notation. Otherwise, explain why it is not polynomial.

Question 2a Indicate whether the following statement is true or false. If it is true give a proof, otherwise give a counter-example.

> Consider a connected graph $G = (V, E)$ with edge weights c_e for all $e \in E$. Let $S \subsetneq V$ be a nonempty node set such that $|\delta(S)| \geq 2$, and let $e \in \delta(S)$ be such that $c_e > c_f$ for all $f \in \delta(S) \setminus \{e\}$. Then, for any minimum spanning tree $T(V, F)$, we have $e \notin F$.

Question 2b Indicate whether the following statement is true or false. If it is true give a proof, otherwise give a counter-example.

> Consider a connected graph $G = (V, E)$ with edge weights c_e for all $e \in E$. Let $e \in E$ be an edge such that $c_e < c_f$ for all $f \in E \setminus \{e\}$. Then, there exists a minimum spanning tree $T(V, F)$ such that $e \in F$.

Question 2c Given a connected graph $G = (V, E)$, with edge costs $c_e > 0, \forall e \in E$, consider the following algorithm:

```
Sort the edges so that c_{e_1} ≥ c_{e_2} ≥ ··· ≥ c_{e_n}
F := E
for i = 1, ..., n
|     if H = (V, F \ {e_i}) is connected
|     |     F := F \ {e_i}
return H = (V, F).
```

Does this algorithm solve the minimum spanning tree problem? If it does, prove that it is correct (i.e., prove that the output is a spanning tree *and* that it has minimum cost). If it does not, give a counterexample (i.e., show an example where the output is not a spanning tree *or* the cost is not minimum).

Question 3a We are given a set $S = \{1, \ldots, n\}$ and some nonempty subset $\emptyset \subsetneq A \subseteq S$. We then define $\mathcal{I} = \{B \subseteq S : A \neq B\}$. Indicate whether or not $(S, \mathcal{I})$ is a matroid. If it is a matroid, show that it satisfies all the necessary conditions for being one. Otherwise, give a counter-example and indicate which condition is not satisfied.

Question 3b We are given a set $S = \{1, \ldots, n\}$ and some nonempty subset $\emptyset \subsetneq A \subseteq S$. We then define $\mathcal{I} = \{B \subseteq S : |B| \leq |A|, A \neq B\}$. Indicate whether or not $(S, \mathcal{I})$ is a matroid. If it is a matroid, show that it satisfies all the necessary conditions for being one. Otherwise, give a counter-example and indicate which condition is not satisfied.

Question 3c Consider a graph $G = (V, E)$ and let $S := V$. Define $\mathcal{I} := \{A \subseteq S : A$ is the set of vertices covered by some matching of $G\}$. Indicate whether or not $(S, \mathcal{I})$ is a matroid. If it is a matroid, show that it satisfies all the necessary conditions for being one. Otherwise, give a counter-example and indicate which condition is not satisfied.

Question 4 For each statement, indicate if the statement is true or false. No justification necessary for this question. In all cases, P and Q are two problems in $\mathcal{NP}$.

1. If both P and Q are $\mathcal{NP}$-complete, then there is a polynomial reduction from P to Q and there is a polynomial reduction from Q to P.

2. If there exist a polynomial reduction from P to Q and a polynomial reduction from Q to P, then we know that both P and Q are $\mathcal{NP}$-complete.

3. if P is $\mathcal{NP}$-complete and there exits a polynomial reduction from Q to P, then Q is $\mathcal{NP}$-complete.

4. if P is in $\mathcal{P}$ and there exits a polynomial reduction from Q to P, then Q is in $\mathcal{P}$.

5. if P is both in $\mathcal{P}$ and in $\mathcal{NP}$, then $\mathcal{P} = \mathcal{NP}$.

Question 5 Solve the following problem using the branch-and-bound method.

$$\begin{array}{rccccccccccl} \max & 2x_1 & + & x_2 & + & 4x_3 & + & 4x_4 & + & 6x_5 & & \\ \text{s.t.} & 7x_1 & + & 21x_2 & + & 28x_3 & + & 19x_4 & + & 17x_5 & \leq 65 \\ & & & & & x \in \{0,1\}^5 & & & & & & \end{array}$$

Hint: The LP relaxation can be solved by sorting the variables according to the ratios c_j/a_j.

Question 6a Consider the following 0-1 knapsack set: $K := \{x \in \{0,1\}^5 : 5x_1+2x_2+3x_3+3x_4+4x_5 \leq 9\}$. Find a cover inequality that separates the fractional point $(1, 1, \frac{1}{3}, \frac{1}{3}, 0)$. Then, use the lifting procedure to strengthen all the cut coefficients that are initially zero.

Question 6b Consider the following 0-1 knapsack set: $K := \{x \in \{0,1\}^5 : 4x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 \leq 10\}$. Find a cover inequality that separates the fractional point $\tilde{x} = \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{6}, 1, 1\right)$. Then, use the lifting procedure to strengthen all the cut coefficients that are initially zero.

Question 7 Find an optimal solution for the following cutting-stock problem. We have a supply of stocks of length 14. We want 20 pieces of length 3 and 18 pieces of length 5. Arrange the cuts so as to satisfy the demand while minimizing the number of stocks used.

Question 8a Consider the following matching problem:

> Given a connected graph $G = (V, E)$, find a subset of the edges $F \subseteq E$ of maximum cardinality, such that no two edges in F are adjacent.

Design a 2-approximation algorithm for the matching problem, and prove that it is indeed a 2-approximation algorithm.
Hint: Use a greedy algorithm.

Question 8b Design a 2-approximation algorithm for weighted vertex cover, and prove that it is indeed a 2-approximation algorithm.
Hint: Use the LP relaxation of an IP formulation for vertex cover.
Remark: You can use without proof any result seen in class, except the proof that the algorithm is a 2-approximation, which you should reproduce here.