University of Waterloo
CO 353 Practice Midterm

Winter 2018

Last name: ______________________________

First name: ______________________________

ID number: ______________________________

Exam details:

Course:	CO 353 – Computational discrete optimization
Duration of exam:	50 minutes
Exam type:	Closed book – no additional material allowed

Instructions:

Your answers must be stated and justified in a clear and logical form, and you must show all of your steps in order to receive full marks. You may use any result from class without proof, unless you are being asked to prove this result. You will be graded not only on correctness, but also on clarity of exposition. No collaboration is allowed.

Question 1 Consider the following algorithms. For each of them, give its computational complexity, in big-O notation, in the arithmetic model. No justification necessary for this question.

1. Given $b \in \mathbb{R}^m, L \in \mathbb{R}^{m \times m}$.
   for $i = 0, 1, \ldots, n-1$
   | $\quad x_i := b_i - \sum_{j=0}^{i-1} L_{ij}$
   return x

2. Given $n \in \mathbb{Z}_+, a, b \in \mathbb{Z}_+^n$.
   $x := 0$
   $y := 0$
   for $i = 0, 1, \ldots, n-1$
   | $\quad$ **for** $j = 0, 1, \ldots, n-1$
   | $\quad$ | $\quad x := x + a_i b_j$
   | $\quad$ | $\quad y := y + \frac{a_i}{b_j + 1}$
   | $\quad$ | $\quad$ **if** $i = j$
   | $\quad$ | $\quad$ | $\quad$ **for** $k = 0, 1, \ldots, n-1$
   | $\quad$ | $\quad$ | $\quad$ | $\quad x := x + a_k y$
   return x

3. Given $x, y \in \mathbb{Z}_+$.

$z := 0$
while $x \neq 0$ **or** $y \neq 0$
| $b := 0$
| **if** x is odd
| | $x := x - 1$
| | $b := 1$
| **if** y is odd
| | $y := y - 1$
| | $b := 1$
| $z := 2z + b$
| $y := y/2$
| $x := x/2$
return z

Question 2 Consider the following problems. For each of them, give its encoding size, in big-O notation, in the bit model. No justification necessary for this question.

1. Given $a, c \in \mathbb{Z}_+^n, b \in \mathbb{Z}_+$,

$$\begin{aligned} \max \quad & \sum_{j=0}^{n-1} c_j x_j \\ \text{s.t.} \quad & \sum_{j=0}^{n-1} a_j x_j \leq b \\ & x \in \mathbb{Z}_+^n, \end{aligned}$$

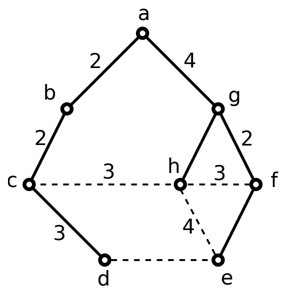
2. Given $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$,

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{R}_+^n, \end{aligned}$$

Question 3 Prove that the statement below is false by giving a counter-example. Draw a counter-example graph G and a subgraph H, and state why H shows that the claim of the theorem is false.

> **Wrong Theorem:** Let $G(V, E)$ be a connected graph and $H(W, F)$ be a subgraph of G. If H is a spanning forest, then $|W| \leq |E| - 2$.

Question 4 Consider the graph $G(V, E)$ illustrated below (continuous and dashed edges), and its subgraph $H(V, F)$ (continuous edges only). The edge costs $c_e > 0$ are given for some edges, but are unknown for the others. Prove that H is not a minimum spanning tree of G. You can use without proof any result seen in class.



Question 5 Let $G(V, E)$ be a connected graph. Consider the ground set $S := E$ and the set $\mathcal{I} := \{J \subseteq E :$ J does not contain any circuit of odd length $\}$. (Circuits of odd length are also called *odd circuits* and graphs who do not contain any are called *bipartite* graphs.) If $(S, \mathcal{I})$ is a matroid, then prove that it satisfies the conditions for being a matroid. Otherwise, if $(S, \mathcal{I})$ is not a matroid, give an example $(S, \mathcal{I})$ instance and show that it does not satisfy one of these conditions.

Question 6 Solve the following integer knapsack problem using the dynamic programming algorithm seen in class.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + 3x_3 \leq 6 \\ & x \in \mathbb{Z}^3_+ \end{aligned}$$

Question 7 Given a decision problem H that is in $\mathcal{NP}$, and another decision problem Q in $\mathcal{P}$, for each statement below, state whether it is (always) true, or (at least in some case) false. No justification necessary.

1. H is not in $\mathcal{P}$.
2. Q is in $\mathcal{NP}$.
3. if Q can be polynomially reduced to H, then H is in $\mathcal{P}$.
4. if H can be polynomially reduced to Q, then H is in $\mathcal{P}$.
5. if H is $\mathcal{NP}$-complete then Q can be polynomially reduced H.