CO 367 Fall 2018: Homework 1

Due: October 5th, 1:30pm at the start of the lecture

Instructions For every nontrivial step you perform, you must justify why the step is valid and what assumption it exploits. In other words, you do not need to justify basic algebraic operations (rearranging or distributing terms, multiplying both sides of an equation by a constant, etc.), but you *do* need to explain all steps that exploit hypotheses and assumptions (positive semidefiniteness of a matrix, continuity or convexity of a function, taking a limit that must exist, etc.).

Question 1 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric, positive semidefinite matrix, and let k be a positive integer.

a. [4 marks] Construct a matrix $G \in \mathbb{R}^{n\times n}$ such that $G^k = A$ for any given A and k. Prove that $G^k = A$.

b. [2 marks] Show that G is invertible if and only if A is invertible.

c. [2 marks] By giving an example, show that G is not necessarily unique in satisfying $G^k = A$.

Question 2 [6 marks] Prove the following. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-smooth function over $\mathbb{R}^n$, and let $\alpha \in \mathbb{R}$ be some constant. If $(\nabla f(y) - \nabla f(x))^T(y - x) \geq \alpha||y - x||_2^2$ for all $x, y \in \mathbb{R}^n$, then $(\nabla^2 f(z) - \alpha I)$ is positive semidefinite for all $z \in \mathbb{R}^n$.

Question 3 Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two convex functions over their respective domains. Moreover, assume that g is monotonically nondecreasing.

a. [3 marks] Show that if we define $h : \mathbb{R}^n \to \mathbb{R}$ as

$$h(x) = g(f(x)),$$

then h is convex over $\mathbb{R}^n$.

b. [3 marks] Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric, positive semidefinite matrix, and let $\beta > 0$ be a positive scalar. Use a. to show that $q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q(x) = e^{\beta x^T Qx}$$

is convex over $\mathbb{R}^n$. (If you use the fact that some function is convex, you must prove that fact.)