CO 367 Fall 2018: Homework 3

Due: ~~December 3rd, 1:30pm~~ – **extended deadline: December 7th, 1:30pm**

Instructions For every nontrivial step you perform, you must justify why the step is valid and what assumptions it exploits. If you exploit a result seen in class, or an elementary theorem, clearly state which one.

Question 1 [4 marks] Find a globally optimal solution to the following problem:

$$\begin{aligned} \min_{x\in\mathbb{R}^3} \quad & c^T x + x^T A x \\ \text{s.t.} \quad & w^T x \geq 4 \\ & x \geq 0, \end{aligned}$$

where $c = [1\ 1\ 1]^T$, $w = [1\ 2\ 1]^T$ and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Justify why this solution is a global minimizer.

Hint 1: First, consider the relaxation of this problem obtained by dropping the $x \geq 0$ constraints. Then, argue that a globally optimal solution for the resulting relaxed problem is globally optimal for the original problem.

Hint 2: You can assume that A is positive definite. One bonus mark for computing its eigenvalues.

Question 2 [3 marks] Let $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = m$, let $t \in \mathbb{R}^n$, and let $b \in \mathbb{R}^m$. Show that we can solve

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & ||x - t||_2 \\ \text{s.t.} \quad & Ax = b, \end{aligned}$$

by just solving a system of linear equations. Justify why your approach yields a global minimizer.

Question 2 [3 marks] Let $Q, R \in \mathbb{R}^{p\times p}$ be two symmetric, positive definite matrices, and let $u, v \in \mathbb{R}^p$. Show that the following problem

$$\begin{aligned} \min_{y,z\in\mathbb{R}^p} \quad & ||y - z||_2 \\ \text{s.t.} \quad & (y-u)^T Q (y-u) \leq 1 \\ & (z-v)^T R (z-v) \leq 1 \end{aligned}$$

can be formulated as a conic optimization problem, i.e., a problem of the form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in K_1 \times \cdots \times K_m, \end{aligned}$$

for some A, b, c and where K_i is one of $\mathbb{R}^{k_i}_+$, $C_2^{k_i+1}$, $S^{k_i}_+$ for all $i = 1, \ldots, m$. You can leave $Ax = b$ in linear constraint notation (no need to construct the matrix A explicitly), but the vector of all variables (x) must belong to a Cartesian product $K_1 \times \cdots \times K_m$ of closed convex cones $\mathbb{R}^{k_i}_+$, $C_2^{k_i+1}$, or $S^{k_i}_+$.