# CO 367 Fall 2018: Homework 3

Due: ~~December 3rd, 1:30pm~~ – extended deadline: December 7th, 1:30pm

**Instructions** For every nontrivial step you perform, you must justify why the step is valid and what assumptions it exploits. If you exploit a result seen in class, or an elementary theorem, clearly state which one.

**Question 1** [4 marks] Find a globally optimal solution to the following problem:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^3} & c^T x + x^T A x \\ \text{s.t.} & w^T x \geq 4 \\ & x \geq 0, \end{array}$$

where $c = [1\ 1\ 1]^T$, $w = [1\ 2\ 1]^T$ and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Justify why this solution is a global minimizer.
*Hint 1:* First, consider the relaxation of this problem obtained by dropping the $x \geq 0$ constraints. Then, argue that a globally optimal solution for the resulting relaxed problem is globally optimal for the original problem.
*Hint 2:* You can assume that $A$ is positive definite. One bonus mark for computing its eigenvalues.

*Solution:* First, we follow the hint and consider the problem:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^3} & c^T x + x^T A x \\ \text{s.t.} & w^T x \geq 4. \end{array}$$

Then, we show that the objective function $f(x) = c^T x + x^T A x$ is convex. Its Hessian is given by $\nabla^2 f(x) = 2A$, and the eigenvalues of $A$ are the roots of

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(2-\lambda)(1-\lambda) - (3-\lambda) - (1-\lambda) \\ &= 6 - 6\lambda - 3\lambda + 3\lambda^2 - 2\lambda + 2\lambda^2 + \lambda^2 - \lambda^3 - 4 + 2\lambda \\ &= -\lambda^3 + 6\lambda^2 - 9\lambda + 2 \\ &= (\lambda - 2)(-\lambda^2 + 4\lambda - 1) \\ &= -(\lambda - 2)(\lambda - (2+\sqrt{3}))(\lambda - (2-\sqrt{3})) \end{aligned}$$

which are all positive. Furthermore, the feasible region is convex because it is a half space (a linear inequality constraint). Therefore, KKT conditions are sufficient for global optimality. Because the feasible region is that of an LP, LPCQ applies, so KKT conditions are also necessary for local (hence global) optimality. Therefore, a point is a global minimizer if and only if it is a KKT point.

We now write the KKT conditions:
- feasibility: $w^T x \geq 4$,
- gradient equation: $-\mu(-w) = 2Ax + c$, where $\mu \geq 0$,
- either $\mu = 0$ or $w^T x = 4$ (or both): $\mu(-w^T x + 4) = 0$. We need to consider two cases.

**First case:** $\mu = 0$.
We get $2Ax = -c$. The inverse of $A$ exists since $A$ is positive definite, and

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix},$$

so

$$x = -\frac{1}{2}A^{-1}c = \begin{bmatrix} -\frac{3}{4} \\ -\frac{7}{4} \\ -\frac{9}{4} \end{bmatrix},$$

which does not satisfy $w^T x \geq 4$ because $w^T x = -\frac{13}{4}$. Therefore, there is no KKT point in this first case.

**Second case:** $w^T x = 4$.
In this case, feasibility is always guaranteed, so we only need to find points that verify $w^T x = 4$ and the gradient equation. From the gradient equation we get $2Ax = \mu w - c$ thus $x = \frac{1}{2}A^{-1}(\mu w - c)$. Then, $w^T x = 4$ becomes

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \left( \mu \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) &= 4 \\ \begin{bmatrix} 1 & \frac{5}{2} & 3 \end{bmatrix} \left( \mu \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) &= 4 \\ \mu \begin{bmatrix} 1 & \frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & \frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= 4 \\ 9\mu - \frac{13}{2} &= 4 \\ \mu &= \frac{7}{6} \end{aligned}$$

We then use the value of $\mu$ in $x = \frac{1}{2}A^{-1}(\mu w - c)$, getting

$$\begin{aligned} x &= \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \left( \frac{7}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \frac{7}{12} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{7}{12} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} \\ \frac{7}{6} \\ \frac{5}{4} \end{bmatrix} \end{aligned}$$

We now have that $x^* = [\frac{5}{12}\ \frac{7}{6}\ \frac{5}{4}]^T$ is the unique global optimal minimizer for

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & c^T x + x^T A x \\ \text{s.t.} \quad & w^T x \geq 4. \end{aligned}$$

Since $x^*$ satisfies $x \geq 0$ and the original problem is a restriction of the one above, $x^*$ is also globally optimal for the original problem.

**Question 2** [3 marks] Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, let $t \in \mathbb{R}^n$, and let $b \in \mathbb{R}^m$. Show that we can solve

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & ||x-t||_2 \\ \text{s.t.} & Ax = b, \end{array}$$

by just solving a system of linear equations. Justify why your approach yields a global minimizer.

*Solution:* Because $x^2$ is monotonously increasing for $x > 0$, $\min_{x\in\Omega} ||x-t||_2$ is equivalent to $\min_{x\in\Omega} ||x-t||_2^2$ which can be written $\min_{x\in\Omega}(x-t)^T(x-t)$, or, by distributing,

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & x^T I x - 2t^T x + t^T t \\ \text{s.t.} & Ax = b, \end{array}$$

We write the KKT conditions:
- feasibility: $Ax = b$,
- gradient equation: $\mu A^T = 2Ix - 2t$, where $\mu \in \mathbb{R}$.

We then have the system of linear equations

$$\begin{bmatrix} A & 0 \\ 2I & -A^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} b \\ 2t \end{bmatrix}.$$

**Question 3** [3 marks] Let $Q, R \in \mathbb{R}^{p\times p}$ be two symmetric, positive definite matrices, and let $u, v \in \mathbb{R}^p$. Show that the following problem

$$\begin{array}{ll} \min\limits_{y,z\in\mathbb{R}^p} & ||y-z||_2 \\ \text{s.t.} & (y-u)^T Q (y-u) \le 1 \\ & (z-v)^T R (z-v) \le 1 \end{array}$$

can be formulated as a conic optimization problem, i.e., a problem of the form

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K_1 \times \cdots \times K_m, \end{array}$$

for some $A, b, c$ and where $K_i$ is one of $\mathbb{R}_+^{k_i}$, $C_2^{k_i+1}$, $S_+^{k_i}$ for all $i = 1, \ldots, m$. You can leave $Ax = b$ in linear constraint notation (no need to construct the matrix $A$ explicitly), but the vector of all variables ($x$) must belong to a Cartesian product $K_1 \times \cdots \times K_m$ of closed convex cones $\mathbb{R}_+^{k_i}$, $C_2^{k_i+1}$, or $S_+^{k_i}$.

*Solution:* Since $Q$ and $R$ are positive definite, there exist matrices $G$ and $H$ such that $Q = GG^T$ and $R = HH^T$.

We can then rewrite

$$
\begin{aligned}
(y-u)^T Q (y-u) &\leq 1 \\
(y-u)^T G G^T (y-u) &\leq 1 \\
s^T s &\leq 1 \\
||s||_2^2 &\leq 1 \\
||s||_2 &\leq 1
\end{aligned}
$$

where $s = G^T y - G^T u$ and similarly $||t||_2 \leq 1$ with $t = H^T z - H^T v$. For the objective function, we can introduce auxiliary variables $w \in \mathbb{R}$ and $x \in \mathbb{R}^n$. We will set $x = y - z$ and enforce $w = ||y-z||_2$ to be our objective function value by minimizing $w$ subject to $w \geq ||y-z||_2 = ||x||_2$.

$$
\begin{aligned}
\min_{w \in \mathbb{R}, x, y, z, s, t \in \mathbb{R}^n} \quad & w \\
\text{s.t.} \quad & x - y + z = 0 \\
& w \geq ||x||_2 \\
& G^T y - s = G^T u \\
& 1 \geq ||s||_2 \\
& H^T z - t = H^T v \\
& 1 \geq ||t||_2
\end{aligned}
$$

We rewrite this as a conic optimization problem by introducing further auxiliary variables $a^1 = a^2 = 1$ and $a^3 = a^4 \in \mathbb{R}$.

$$
\begin{aligned}
\min \quad & w \\
\text{s.t.} \quad & a^1 = 1 \\
& a^2 = 1 \\
& x - y + z = 0 \\
& G^T y - s = G^T u \\
& H^T z - t = H^T v \\
& (a^1, s) \in C_2^{n+1} \\
& (a^2, t) \in C_2^{n+1} \\
& (w, x) \in C_2^{n+1} \\
& (a^3, y) \in C_2^{n+1} \\
& (a^4, z) \in C_2^{n+1}.
\end{aligned}
$$

Note that we needed to add the free variables $a^3$ and $a^4$ so that all our variables, including $y$ and $z$ belong to some cone.