CO 367 Fall 2018: Numerical exercises

Note: If you feel that some numerical exampes can help you build intuition about convex functions and their relationship to positive semidefinite matrices, feel free to try these exercises. **Midterm questions will be theoretical and will not be similar to the exercises given here in any way.**

Question 1 Consider the quadratic function $q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$q(x) = x_1^2 + 2x_2^2 + 3x_3^2 + 6x_1x_2 - 2x_1x_3 - 4x_2x_3$$

(a) Determine a matrix Q such that $q(x) = x^T Qx$.

(b) Determine $a_1, a_2, a_3 \in \mathbb{R}$ and a transformation matrix $G \in \mathbb{R}^{3\times 3}$ such that $q(x) = f(Gx)$ and $f(x) = a_1x_1^2 + a_2x_2^2 + q_3x_3^2$.

(c) Show that neither Q nor $-Q$ is positive semidefinite. In particular, find $y, z \in \mathbb{R}^3$ such that $y^T Qy > 0$ and $z^T Qz > 0$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(\lambda) = q(\lambda y)$. Show that f is convex.

(e) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(\sigma) = q(\sigma(y + z))$. Show that g is not convex.

Question 2 Let

$$Q = \begin{bmatrix} 5 & 3 & 2 \\ -3 & 9 & -1 \\ -2 & -3 & 6 \end{bmatrix}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = x^T Qx$.

(a) Find an expression for $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ as a function of x_1, x_2, x_3.

(b) Find an expression for $\nabla^2 f : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ as a function of x_1, x_2, x_3.

(c) Find a symmetric matrix A such that $x^T Qx = x^T Ax$ for all $x \in \mathbb{R}^3$.

(d) Compute the eigenvalues and eigenvectors v_1, v_2, v_3 of A.

(e) Let $g_i : \mathbb{R} \to \mathbb{R}$ be given by $g_i(t) = q(t \cdot v_i)$. Give an expression for g_i as a function of t, for all i.

(f) Determine whether or not f is convex over $\mathbb{R}^3$.