CO 370 Fall 2019: Homework 2

Due: November 1st by 1:00pm

Instructions You will be graded not only on correctness, but also on clarity of exposition. You are allowed to talk with classmates about the assignment as long as (1) you acknowledge the people you collaborate with, (2) you write your solutions on your own, and (3) you are able to fully explain your solutions. In the models, always give a clear definition to your decision variables (in most cases, this means that you must explain what they represent in plain words). In case you run into trouble (a question is ambiguous, data provided have an issue, problem with the implementation, etc.), it is your responsibility to ask me or your TAs for clarifications in a timely manner.

Homework submission Questions 1, 2 and 3 are to be submitted on Crowdmark. The model file for Question 4 is to be submitted on Learn.

Question 1 [3 marks] Consider the following linear programming problem

min [redacted]

$$
\begin{array}{rrrrrl}
\text{s.t.} & 2x_1 + & x_2 + x_3 + & x_4 = & 5 \\
& 2x_1 + 3x_2 & + & x_4 \leq & 12 \\
& x_1 - & x_2 + x_3 & \leq & 5 \\
& & & x_4 \geq & -5 \\
& & x_2 & - x_4 \geq & 4 \\
& 2x_1 + 2x_2 - & x_3 - & x_4 \geq & 2 \\
& x_1 \;, & x_2 \;, x_3 \;, & x_4 & \text{free} \;.
\end{array}
\qquad \text{(P)}
$$

The objective function has been censored, but you know that a dual optimal solution is given by

$$y^* = \left(2, \; -\frac{1}{2}, \; 0, \; 5, \; \frac{11}{2}, \; 0\right).$$

a. Determine a (primal) optimal solution x^* for (P).

Solution: Observe that y_1^*, y_2^*, y_4^* and y_5^* are nonzero. By complementary slackness, we thus know that the first, second, fourth and fifth constraint of (P) are tight at x^*. This gives us the system

$$
\begin{cases}
2x_1 + x_2 + x_3 + x_4 = 5 \\
2x_1 + 3x_2 + x_4 = 12 \\
x_4 = -5 \\
x_2 - x_4 = 4
\end{cases}
$$

We immediately obtain $x_4 = -5$. By substitution, we then get $x_2 = -1$, $x_1 = 10$ and $x_3 = -9$, yielding

$$x^* = (10, \; -1, \; -9, \; , -5)\,.$$

b. Denoting the objective function of (P) by

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4,$$

write the dual (D) of (P).

Solution:

$$
\begin{array}{lllllllllllll}
\max & 5y_1 & + & 12y_2 & + & 5y_3 & - & 5y_4 & + & 4y_5 & + & 2y_6 & \\
\text{s.t.} & 2y_1 & + & 2y_2 & + & y_3 & & & & & + & 2y_6 & = c_1 \\
& y_1 & + & 3y_2 & - & y_3 & & & + & y_5 & + & 2y_6 & = c_2 \\
& y_1 & & & + & y_3 & & & & & - & y_6 & = c_3 \\
& y_1 & + & y_2 & & & + & y_4 & - & y_5 & - & y_6 & = c_4 \\
& \multicolumn{12}{l}{y_1 \text{ free}, \quad y_2 \le 0, \quad y_3 \le 0, \quad y_4 \ge 0, \quad y_5 \ge 0, \quad y_6 \ge 0,}
\end{array} \tag{D}
$$

c. Determine values for c_1, c_2, c_3, c_4 such that x^* is optimal for (P) and y^* is optimal for (D).

Solution: We simply plug y^* in the constraints of (D), yielding

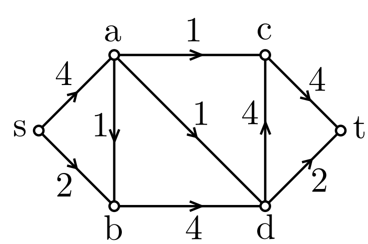
$$
\begin{aligned}
2y_1 + 2y_2 + y_3 \qquad\qquad + 2y_6 &= c_1 = 3 \\
y_1 + 3y_2 - y_3 \qquad + y_5 + 2y_6 &= c_2 = 6 \\
y_1 \qquad + y_3 \qquad\qquad - y_6 &= c_3 = 2 \\
y_1 + y_2 \qquad + y_4 - y_5 - y_6 &= c_4 = 1
\end{aligned}
$$

i.e.,

$$c = (3,\ \ 6,\ \ 2,\ \ 1)\,.$$

Question 2 [4 marks]

a. Consider the following directed graph $G = (V, E)$ with capacities $u_e \ge 0$ for all $e \in E$



and the corresponding max s-t flow formulation

$$
\begin{array}{lllllllllll}
\max & x_{sa} + x_{sb} & & & & & & & & & \\
\text{s.t.} & -x_{sa} & + x_{ab} + x_{ac} + x_{ad} & & & & & & = 0 & (a) \\
& - x_{sb} - x_{ab} & & + x_{bd} & & & & & = 0 & (b) \\
& & - x_{ac} & & - x_{dc} + x_{ct} & & & & = 0 & (c) \\
& & - x_{ad} - x_{bd} + x_{dc} & & & + x_{dt} & & & = 0 & (d) \\
\multicolumn{11}{l}{x_{sa} \le 4,\ x_{sb} \le 2,\ x_{ab} \le 1,\ x_{ac} \le 1,\ x_{ad} \le 1,\ x_{bd} \le 4,\ x_{dc} \le 4,\ x_{ct} \le 4,\ x_{dt} \le 2,} \\
\multicolumn{11}{l}{x_{sa} \ge 0,\ x_{sb} \ge 0,\ x_{ab} \ge 0,\ x_{ac} \ge 0,\ x_{ad} \ge 0,\ x_{bd} \ge 0,\ x_{dc} \ge 0,\ x_{ct} \ge 0,\ x_{dt} \ge 0.}
\end{array}
$$

Write the dual of the above LP. All the constraints of the dual must be written individually (i.e., not in matrix notation).

Solution: We introduce a dual variable y_q for all $q \in V \setminus \{s, t\}$, corresponding to the $= 0$ constraints, and a variable z_{qr} for all $qr \in E$, corresponding to the $\leq u_{qr}$ bounds. The dual is given by

$$
\begin{array}{llllllll}
\min & & & & & 4z_{sa} + 2z_{sb} + z_{ab} + z_{ac} + z_{ad} + 4z_{bd} + 4z_{dc} + 4z_{ct} + 2z_{dt} & & \\
\text{s.t.} & -y_a & & & & + z_{sa} & \geq & 1 \\
& & - y_b & & & + z_{sb} & \geq & 1 \\
& y_a & - y_b & & & + z_{ab} & \geq & 0 \\
& y_a & & - y_c & & + z_{ac} & \geq & 0 \\
& y_a & & & - y_d & + z_{ad} & \geq & 0 \\
& & y_b & & - y_d & + z_{bd} & \geq & 0 \\
& & & - y_c & + y_d & + z_{dc} & \geq & 0 \\
& & & y_c & & + z_{ct} & \geq & 0 \\
& & & & y_d & + z_{dt} & \geq & 0 \\
& y_a & , y_b & , y_c & , y_d & \text{free} & & \\
& & & & & z_{sa}, z_{sb}, z_{ab}, z_{ac}, z_{ad}, z_{bd}, z_{dc}, z_{ct}, z_{dt} & \geq & 0.
\end{array}
$$

b. More generally, consider any directed graph $G = (V, E)$, and any capacities $u_{qr} \geq 0$ for all $qr \in E$. The max s-t flow formulation we have seen in class is

$$
\begin{array}{lllll}
\max & \sum_{r\,:\,sr \in E} x_{sr} & & & \\
\text{s.t.} & \sum_{r\,:\,qr \in E} x_{qr} - \sum_{r\,:\,rq \in E} x_{rq} & = & 0, & \forall q \in V \setminus \{s,t\} \\
& 0 \leq x_{qr} \leq u_{qr}, & & & \forall qr \in E.
\end{array}
$$

Write the dual of this LP. All the constraints of the dual must be written explicitly (i.e., not in matrix notation).

Solution: Like for the previous subquestion, we introduce a dual variable y_q for all $q \in V \setminus \{s,t\}$, corresponding to the $= 0$ constraints, and a variable z_{qr} for all $qr \in E$, corresponding to the $\leq u_{qr}$ bounds. We obtain the following dual:

$$
\begin{array}{lllllll}
\min & & & \sum_{qr \in E} u_{qr} z_{qr} & & & \\
\text{s.t.} & & - y_r & + z_{sr} & \geq & 1 & \forall r : sr \in E \\
& y_q & & + z_{qt} & \geq & 0 & \forall q : qt \in E \\
& y_q & - y_r & + z_{qr} & \geq & 0 & \forall q, r \in V \setminus \{s,t\} : qr \in E \\
& y_q & \text{free} & & & & \forall q \in V \setminus \{s,t\} \\
& & & z_{qr} & \geq & 0 & \forall qr \in E.
\end{array}
$$

Question 3 [4 marks] We want to decide the best possible position $x \in \mathbb{R}^2$ to place a 5G antenna. The ideal position is a known constant $b \in \mathbb{R}^2$. Unfortunately, we cannot place the antenna there, so we want to minimize the 1-norm distance

$$||x - b||_1 = |x_1 - b_1| + |x_2 - b_2|$$

instead. The set of possible positions is given as a list of triangles: the coordinates (x_1, x_2) of the antenna must lie in (at least) one of the triangles. The triangles are described by their vertices

$$v^{t,1}, v^{t,2}, v^{t,3} \in \mathbb{R}^2, \qquad \text{for } t = 1, \ldots, T,$$

where we have T triangles in total. Formulate the problem of finding the best possible position x as an integer programming problem.

Solution:

Variables:

$$
\begin{aligned}
x_j &= \text{coordinates of the antenna position, } j = 1, 2 \\
d_j &= |x_j - b_j|, \ j = 1, 2 \\
z_t &= \begin{cases} 1 \text{ if } x \text{ is constructed as a point of triangle } t, \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } t = 1, \ldots, T \\
\lambda^{t,i} &= \text{multiplier for vertex } i \text{ of triangle } t, \ \text{for } t = 1, \ldots, T, \ i = 1, 2, 3.
\end{aligned}
$$

Model:

$$
\begin{aligned}
\min \quad & d_1 + d_2 \\
& -d_j \le x_j - b_j \le d_j && \text{for } j = 1, 2 \\
& x_j = \sum_{t=1}^{T} \sum_{i=1}^{3} \lambda^{t,i} v_j^{t,i}, && \text{for } j = 1, 2 \\
& 1 = \sum_{t=1}^{T} \sum_{i=1}^{3} \lambda^{t,i} \\
& 0 \le \lambda^{t,i} \le z_t && \text{for } t = 1, \ldots, T, \ i = 1, \ldots, 3 \\
& \sum_{t=1}^{T} z_t = 1 \\
& z_t \in \{0, 1\} && \text{for } t = 1, \ldots, T
\end{aligned}
$$

The first constraint $-d_j \le x_j - b_j \le d_j$ takes care of ensuring $d_j \ge |x_j - b_j|$. Since we are minimizing d_j, we will have $d_j = |x_j - b_j|$ which allows us to have $||x - b||_1 = d_1 + d_2$.

The second constraint

$$x_j = \sum_{t=1}^{T} \sum_{i=1}^{3} \lambda^{t,i} v_j^{t,i}$$

lets x be a linear combination of the vertices of one of the triangles, as long as we ensure that $\lambda^{t,i} = 0$ for all i and for all triangles t except one. This is enforced by

$$\lambda^{t,i} \le z_t \quad \text{and} \quad \sum_{t=1}^{T} z_t = 1.$$

Finally,

$$0 \le \lambda^{t,i} \quad \text{and} \quad 1 = \sum_{t=1}^{T} \sum_{i=1}^{3} \lambda^{t,i}$$

ensure that the linear combination is actually a convex combination.

Note that, as an alternative solution, we could have combined the above two into

$$0 \le \lambda^{t,i} \quad \text{and} \quad z_t = \sum_{i=1}^{3} \lambda^{t,i} \quad \text{for all } t,$$

yielding the model

$$\begin{aligned}
\min \quad & d_1 + d_2 \\
& -d_j \le x_j - b_j \le d_j && \text{for } j = 1, 2 \\
& x_j = \sum_{t=1}^{T} \sum_{i=1}^{3} \lambda^{t,i} v_j^{t,i}, && \text{for } j = 1, 2 \\
& z_t = \sum_{i=1}^{3} \lambda^{t,i} && \text{for } t = 1, \ldots, T \\
& 0 \le \lambda^{t,i} && \text{for } t = 1, \ldots, T,\ i = 1, \ldots, 3 \\
& \sum_{t=1}^{T} z_t = 1 \\
& z_t \in \{0, 1\} && \text{for } t = 1, \ldots, T
\end{aligned}$$

Question 4 [4 marks] Implement the model of Question 3 in Julia.

You nust follow the specifications given here exactly. At least one mark will be automatically subtracted if your code does not fully follow the specifications. If your code cannot be run after minor modifications, you will get 0 marks for the question.

Carefully check the following:

- Your model must be uploaded in a single Julia file (for example `hw2-model.jl`).
- Your model file must be uploaded to Learn by the project deadline.
- The data will be given to your model in the form of a data file to be included before including your model. Your model must still work if the data is changed (it will be tested with multiple variants of the data file). Therefore, the model file must not itself `include("")` a data file.
- Your model file is responsible for solving the model. In other words, it must contain a call to `optimize!()`.
- After the model file is run, an array named `xs` must contain an optimal solution. Note that `xs` thus cannot be the name of a model variable. Instead, you would have something similar to `xs = value.(x)` at the end of your model file, if `x` is the name of some model variable.

The following table gives the correspondence between mathematical notation and Julia names:

	Mathematical notation	Julia notation
ideal placement	b_1, b_2	`b[1]`, `b[2]`
triangle $t = 1, \ldots, T$, vertex $i = 1, 2, 3$, coordinate $j = 1, 2$	$v_j^{t,i}$	`v[t, i, j]`
optimal solution	x_1^*, x_2^*	`xs[1]`, `xs[2]`

An example data file can be downloaded here:
`https://www.math.uwaterloo.ca/~lpoirrie/co370/d/hw2-data-00.jl`
You can plot the data and solution using this script:
`https://www.math.uwaterloo.ca/~lpoirrie/co370/d/hw2-check.jl`

Solution:

```
using JuMP
using Cbc

md = Model(with_optimizer(Cbc.Optimizer))

@variable(md, x[1:2])
@variable(md, d[1:2] >= 0)
@variable(md, z[1:T], Bin)
@variable(md, 0 <= λ[1:T, 1:3] <= 1)

@objective(md, Min, d[1] + d[2])

# absolute value
@constraint(md, [j in 1:2],
     -d[j] <= x[j] - b[j])

@constraint(md, [j in 1:2],
     x[j] - b[j] <= d[j])

# convex combination
@constraint(md, [j in 1:2],
     x[j] == sum(λ[t, i] * v[t, i, j] for t in 1:T, i in 1:3))

@constraint(md,
     sum(λ[t, i] for t in 1:T, i in 1:3) == 1)

@constraint(md, [t in 1:T, i in 1:3],
     λ[t, i] <= z[t])

@constraint(md, sum(z[t] for t in 1:T) == 1)

optimize!(md)

xs = value.(x)
```