CO 370 Fall 2019: Homework 3

Due: November 26 by 2:00pm

Instructions You will be graded not only on correctness, but also on clarity of exposition. You are allowed to talk with classmates about the assignment as long as (1) you acknowledge the people you collaborate with, (2) you write your solutions on your own, and (3) you are able to fully explain your solutions. In the models, always give a clear definition to your decision variables (in most cases, this means that you must explain what they represent in plain words). In case you run into trouble (a question is ambiguous, data provided have an issue, problem with the implementation, etc.), it is your responsibility to ask me or your TAs for clarifications in a timely manner.

Homework submission Your solutions are to be submitted on Crowdmark.

Question 1 [12 marks]

Consider the following linear programming problem (P) in SEF.

$$\begin{array}{lrcrcrcrcrcl}
\min & -x_1 & - & 2x_2 & & & & & & & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 6 \\
 & & - & x_2 & & & & & + & x_5 & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

(a) Prove that the basis $\mathcal{B} = \{2, 4, 5\}$ is an optimal basis, and write down the corresponding optimal basic solution.

(b) Find the allowable range for the cost c_j of every variable x_j (for $j = 1, \dots, 5$). In other words, find the range of values that θ can take such that $\mathcal{B}$ remains an optimal basis when we replace the objective function coefficient c_j by $c_j + \theta$ (note: only one objective coefficient is changing at a time).

(c) Find the allowable range for the right-hand side b_i of each constraint i (for $i = 1, \dots, 3$) other than the non-negativity constraints. In other words, find the range of values that θ can take such that $\mathcal{B}$ remains an optimal basis when we change the right-hand side b_i to $b_i + \theta$ (note: only one right-hand side is changing at a time).

(d) Suppose that the right-hand side of the second constraint changes from 6 to 3. Find a new optimal solution by applying the dual simplex algorithm.

Solution:

(a) The basis $\mathcal{B} = \{2, 4, 5\}$ is optimal if the basic solution is nonnegative and the corresponding reduced costs are nonnegative. We can either apply the formulas $\bar{x}_{\mathcal{B}} = \bar{b} = B^{-1}b$ and $\bar{c}^T = c^T - c_{\mathcal{B}}^T B^{-1} A$ or compute the full tableau corresponding to $\mathcal{B}$. To get the tableau, we first express the basic variables as a function of the nonbasic variables:

$$\begin{array}{rcrcrcr} x_2 & = & 4 & - & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & + & x_3 \\ x_5 & = & 7 & - & x_1 & - & x_3 \end{array} .$$

We can now eliminate x_1, x_2 and x_4 from the objective function, yielding the tableau

$$\begin{array}{lccccccccccc} \min & x_1 & & & + & 2x_3 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\ & x_1 & & & - & x_3 & + & x_4 & & & = & 2 \\ & x_1 & & & + & x_3 & & & + & x_5 & = & 7 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

Since $\bar{x} = (0, 4, 0, 2, 7) \geq 0$ and $\bar{c} = (1, 0, 2, 0, 0) \geq 0$, the basis $\mathcal{B} = \{2, 4, 5\}$ is optimal.

(b) We computed $\bar{c}$ in part (a). For the nonbasic variables we simply need to have $\bar{c}_j + \theta \geq 0$. Thus, for x_1, we need $1 + \theta \geq 0$ and for x_3, we need $2 + \theta \geq 0$.

For the basic variables, we need $\bar{c}_{\mathcal{N}}^T - \theta \cdot e_i^T B^{-1} N \geq 0$ where $e_i^T B^{-1} N$ is the nonbasic part of the ith row of the tableau above. For x_2, which is basic in row $i = 1$, this means

$$[1 \quad 2] - \theta [1 \quad 1] \geq 0, \qquad \text{thus } \theta \leq 1 \text{ (and } \theta \leq 2).$$

For x_4, which is basic in row $i = 2$, this means

$$[1 \quad 2] - \theta [1 \quad -1] \geq 0, \qquad \text{thus } \theta \leq 1 \text{ and } \theta \geq -2.$$

For x_5, which is basic in row $i = 3$, this means

$$[1 \quad 2] - \theta [1 \quad 1] \geq 0, \qquad \text{thus } \theta \leq 1 \text{ (and } \theta \leq 2).$$

In summary, we have

$$\begin{array}{lrl} c_1' = c_1 + \theta & \text{nonbasic} & \theta \in [-1; +\infty] \\ c_2' = c_2 + \theta & \text{basic} & \theta \in [-\infty; +1] \\ c_3' = c_3 + \theta & \text{nonbasic} & \theta \in [-2; +\infty] \\ c_4' = c_4 + \theta & \text{basic} & \theta \in [-2; +1] \\ c_5' = c_5 + \theta & \text{basic} & \theta \in [-\infty; +1] \end{array}$$

(c) For b_i, we require $B^{-1} b + \theta \cdot B^{-1} e_i \geq 0$, i.e. $\bar{b} + \theta \cdot B^{-1} e_i \geq 0$, where $B^{-1} e_i$ is the ith column of B^{-1}. Since the last three columns of A are an identity matrix, the relation $\bar{A} = B^{-1} A$ gives us $\bar{A}_{\{3,4,5\}} = B^{-1} I$, giving us B^{-1} immediately:

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

For b_1 we have

$$\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + \theta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \geq 0, \text{ thus } \left\{ \begin{array}{lcr} \theta & \geq & -4 \\ \theta & \leq & 2 \\ \theta & \geq & -7. \end{array} \right.$$

For b_2 we have

$$\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \geq 0, \text{ thus } \theta \geq -2.$$

For b_3 we have

$$\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \geq 0, \text{ thus } \theta \geq -7.$$

In summary, we have

$$\begin{aligned} b_1' &= b_1 + \theta, \quad \theta \in [-4; +2] \\ b_2' &= b_2 + \theta, \quad \theta \in [-2; +\infty] \\ b_3' &= b_3 + \theta, \quad \theta \in [-7; +\infty] \end{aligned}$$

(d) We compute the new right-hand sides $\bar{b}'$ for the tableau associated with the (previously-optimal) basis $\{2, 4, 5\}$. We have

$$\bar{b}' = B^{-1}b' = B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}.$$

We thus have the starting tableau

$$\begin{array}{rrrrrrrrrrrr} \min & x_1 & & & + & 2x_3 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\ & x_1 & & & - & x_3 & + & x_4 & & & = & -1 \\ & x_1 & & & + & x_3 & & & + & x_5 & = & 7 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

In the dual simplex method, we need all reduced-costs to be nonnegative (which is the case) and we pick a row with a negative right-hand side to determine the leaving variable. Row $i = 2$ has $\bar{b}_2 = -1$, so x_4, which is basic in row 2, will leave the basis. The ratio test is $\min\{\frac{\bar{c}_j}{A_{ij}} | A_{ij} < 0\}$. We only have one $A_{ij} < 0$, namely $A_{23} = -1$ for $j = 3$ so x_3 will enter the basis. Our next basis is thus $\{2, 3, 5\}$ and we obtain the following corresponding tableau:

$$\begin{array}{rrrrrrrrrrrrr} \min & & 3x_1 & & & & & + & 2x_4 & & & & \\ \text{s.t.} & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 3 \\ & - & x_1 & & & + & x_3 & - & x_4 & & & = & 1 \\ & & 2x_1 & & & & & + & x_4 & + & x_5 & = & 6 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

We see that all basic variables are ≥ 0, so this basis is actually optimal. The new optimal solution is $x^* = (0, 3, 1, 0, 6)$ with objective function value $z^* = -x_1^* - 2x_2^* = -6$.

Question 2 [10 marks]

A company produces 4 types of products P_1, P_2, P_3, P_4 using 4 types of resources R_1, R_2, R_3, R_4. The following table shows the amount of each resource that is needed to produce one unit of each product, together with the total availability for each resource and the profit obtainable by selling one unit of each product.

	P_1	P_2	P_3	P_4	resource availability
R_1	10	15	20	20	130
R_2	1	2	3	1	13
R_3	3	1	12	3	45
R_4	2	4	7	3	23
profit ($)	51	102	132	89	

(a) Formulate as an LP the problem of deciding the production plan in order to maximize the profit.

Solution:

$$
\begin{array}{lrcrcrcrcr}
\max & 51x_1 & + & 102x_2 & + & 132x_3 & + & 89x_4 & & \\
\text{s.t.} & 10x_1 & + & 15x_2 & + & 20x_3 & + & 20x_4 & \leq & 130 \\
 & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 13 \\
 & 3x_1 & + & x_2 & + & 12x_3 & + & 3x_4 & \leq & 45 \\
 & 2x_1 & + & 4x_2 & + & 7x_3 & + & 3x_4 & \leq & 23 \\
 & & & & & & & x & \geq & 0
\end{array}
$$

(b) Suppose that the shadow prices and the used amount of each resource in an optimal solution are as follows:

resource	shadow price	used amount
R_1	1.429	130
R_2	0	9
R_3	0	17
R_4	20.143	23

(i) The company has the option to sell 20 units of the resource R_1 at 1.1 dollar per unit to another company. Without solving a new LP, state whether or not the company should go for this option. Justify your answer.

Solution: The company should not go for this option. Reducing the availability of R_1 by 20 units will decrease the objective function value by at least $20 \cdot 1.429 = 28.58$ dollars which is more than $20 \cdot 1.1 = 22$ dollars that the company is getting from selling the resource to another company.

(ii) The company has the possibility to produce a new product P_5 and sell it for 10 dollars per unit. To produce 1 unit of the new product the company needs 2 units of R_2 and 4 units of R_3. Without solving a new LP, state whether or not the company should consider this option. Justify your answer.

Solution: The company should consider this option. Note that the shadow prices for constraints 2 and 3 are zero, and one can see that reducing the availability of R_2 by 2 units and of R_3 by 4 units will not decrease the objective function value (we are not using the whole available amount of these resources), while the company could increment the profit by selling a positive amount of P_5.

(iii) The company has the possibility to produce a new product P_6. To produce 1 unit of the new product the company needs 3 units of R_1, 2.4 units of R_2 and 5.1 units of R_4. What should be the minimum price that the company should set for 1 unit of P_6 in order to consider this option? Justify your answer.

Solution: Diverting 3 units of R_1, 2.4 units of R_2 and 5.1 units of R_4 to produce P_6 will decrease the objective function value by at least $3 \cdot 1.429 + 5.1 \cdot 20.143 = 107.0163$ dollars, and therefore the price of P_6 should be at least that amount to consider the option.

(iv) The company has the possibility to sell to another company *any* amount of resource R_4 at a price of 20 dollars per unit. How does this change the model? Without solving a new LP, can you say whether the current optimal solution will still be optimal for the new model?

Solution: This can be modelled by introducing a slack variable for constraint 4 with objective function coefficient equal to 20. According to the data we have, diverting any amount α of resource R_4 will decrease the objective function by at least 20.143α, and the company will get only 20α from selling the resource to another company. So, even if the company has this option, the current solution is still optimal.

Question 3 [10 marks]

Consider the linear programming problem

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0, \end{array} \tag{P}$$

where $A \in \mathbb{R}^{3\times 5}$, $b \in \mathbb{R}^3$ and $c \in \mathbb{R}^5$. We know that the last three columns of A form an identity matrix, i.e.

$$A = \begin{bmatrix} ? & ? & 1 & 0 & 0 \\ ? & ? & 0 & 1 & 0 \\ ? & ? & 0 & 0 & 1 \end{bmatrix}.$$

We are given a tableau of (P) corresponding to the basis $\mathcal{B} = \{2, 4, 1\}$:

$$\begin{array}{lccccccccccc} \min & & & & & \bar{c}_3 x_3 & & & + & \bar{c}_5 x_5 & & \\ \text{s.t.} & & & x_2 & - & x_3 & & & + & \beta x_5 & = & 1 \\ & & & & & 2x_3 & + & x_4 & + & \gamma x_5 & = & 2 \\ & x_1 & & & + & 4x_3 & & & + & \delta x_5 & = & 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0, \end{array}$$

where $\bar{c}_3, \bar{c}_5, \beta, \gamma, \delta \in \mathbb{R}$ are constants.

1. Give (necessary and sufficient) conditions under which $\mathcal{B}$ is an optimal basis.
2. Suppose that $\mathcal{B}$ is optimal and $\bar{c}_3 = 0$. Let $\tilde{x}$ be the basic solution associated to $\mathcal{B}$. Find a basic feasible solution that is also optimal but distinct from $\tilde{x}$.
3. Suppose that $\gamma > 0$. Show that there always exists a (finite) optimal solution, regardless of the values of $\bar{c}_3$ and $\bar{c}_5$.
4. Suppose that $\mathcal{B}$ is optimal. Give the allowable range for b_1, the right-hand side of the first constraint in the initial problem. In other words, give the values that θ can take such that $\mathcal{B}$ remains optimal when b_1 is replaced by $b_1 + \theta$. (Beware that we want the range on the right-hand side b_1 of the *initial* problem, whose numerical value is not given, *not* the range on $\bar{b}_1 = 1$ in the optimal tableau corresponding to $\mathcal{B}$.) Parameters can appear in the answer.
5. Suppose that $\mathcal{B}$ is optimal. Give the allowable range for c_1, the cost of x_1 in the initial problem. In other words, give the values that θ can take such that $\mathcal{B}$ remains optimal when c_1 is replaced by $c_1 + \theta$. (Beware

that we want the range on the cost c_1 of the *initial* problem, whose numerical value is not given, *not* the range on $\bar{c}_1 = 0$ in the optimal tableau corresponding to $\mathcal{B}$.) Parameters can appear in the answer.

Solution:

1. The basis $\mathcal{B}$ is optimal if and only if $\bar{c}_3 \geq 0$ and $\bar{c}_5 \geq 0$

2. We have $\tilde{x} = (3,\ 1,\ 0,\ 2,\ 0)$. We can let x_3 enter the basis, yielding another solution with the same cost. The ratio test (of the primal simplex method) gives us:

$$\min\{\frac{2}{2}, \frac{3}{4}\} = \frac{3}{4}$$

so x_1 leaves the basis. The new tableau is

$$\begin{array}{llllllll}
\min & & & & & \bar{c}_5 x_5 & & \\
\text{s.t.} & \frac{1}{4}x_1 & +x_2 & & & +(\beta + \frac{\delta}{4})x_5 & = & 7/4 \\
& -\frac{1}{2}x_1 & & & +x_4 & +(\gamma - \frac{\delta}{2})x_5 & = & 1/2 \\
& \frac{1}{4}x_1 & & +x_2 & & +(\frac{\delta}{4})x_5 & = & 3/4 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0
\end{array}$$

Another optimal basic solution is thus

$$x = (0,\ 7/4,\ 3/4,\ 1/2,\ 0).$$

3. For this subquestion, let us consider the problem (P') given by the tableau associated with basis $\{2, 4, 1\}$, instead of the original problem (P). Indeed, since tableaux are just reformulations of the original problem, (P) has an optimal solution if and only if (P') has one. From now on, we will refer to (P') as our primal.

We will show that there exist both a primal feasible solution and a dual feasible solution. Thus, by weak duality, the primal is feasible and bounded below. As a consequence an optimal solution must exist.

A primal feasible solution is given by, for example, $\tilde{x} = (3,\ 1,\ 0,\ 2,\ 0)$.

The dual of (P') is

$$\begin{array}{lccccccl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
\text{s.t.} & & & & & y_3 & \leq & 0 \\
& y_1 & & & & & \leq & 0 \\
& -y_1 & + & 2y_2 & + & 4y_3 & \leq & \bar{c}_3 \\
& & & y_2 & & & \leq & 0 \\
& \beta y_1 & + & \gamma y_2 & + & \delta y_3 & \leq & \bar{c}_5 \\
& y_1 & , & y_2 & , & y_3 & & \text{free}
\end{array}$$

which simplifies to

$$\begin{array}{lccccccl}
\max & y_1 & + & 2y_2 & + & 3y_3 & & \\
\text{s.t.} & -y_1 & + & 2y_2 & + & 4y_3 & \leq & \bar{c}_3 \\
& \beta y_1 & + & \gamma y_2 & + & \delta y_3 & \leq & \bar{c}_5 \\
& y_1 & , & y_2 & , & y_3 & \leq & 0.
\end{array}$$

Let us try to construct a feasible solution. We start by fixing as many variables as possible to zero to simplify the construction. In particular, if we fix $y_1 = 0$ and $y_3 = 0$, then we need to find a value of y_2 such that

$$\begin{aligned} 2y_2 &\leq \bar{c}_3 \\ \gamma y_2 &\leq \bar{c}_5 \\ y_2 &\leq 0. \end{aligned}$$

Let us consider

$$\mu = \min\left\{\frac{\bar{c}_3}{2}, \frac{\bar{c}_5}{\gamma}, 0\right\}.$$

We know that μ exists because $\gamma \neq 0$. We now have that $\tilde{y} = (0, \mu, 0)$ is dual feasible.

4. Since $A = [A_1\ A_2\ I]$, the simplex tableau gives us $\bar{A} = B^{-1}A = B^{-1}[A_1\ A_2\ I]$, so we have

$$B^{-1} = \begin{bmatrix} -1 & 0 & \beta \\ 2 & 1 & \gamma \\ 4 & 0 & \delta \end{bmatrix}.$$

We want $x'_{\mathcal{B}} \geq 0$, where

$$x'_{\mathcal{B}} = B^{-1}(b + \theta e_1) = B^{-1}b + B^{-1}\theta e_1$$

thus

$$B^{-1}b + \theta \cdot B^{-1}e_1 \geq 0$$

$$\Leftrightarrow \begin{bmatrix} -\theta \\ 2\theta \\ 4\theta \end{bmatrix} \geq \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \Leftrightarrow \begin{cases} \theta \leq 1 \\ \theta \geq -1 \\ \theta \geq -3/4 \end{cases} \Leftrightarrow -3/4 \leq \theta \leq 1$$

5. x_1 is basic in the row $i = 3$, so the optimality condition can be expressed

$$\begin{aligned} &\bar{c}'^T = c'^T - c_B'^T B^{-1}A \geq 0 \\ &\Rightarrow c^T + \theta e_1^T - (c_B^T + \theta e_3^T)B^{-1}A \geq 0 \\ &\Leftrightarrow c^T - c_B^T B^{-1}A + \theta e_1^T - \theta e_3^T B^{-1}A \geq 0 \\ &\Leftrightarrow c^T - c_B^T B^{-1}A + \theta e_1^T - \theta[1\ 0\ 4\ 0\ \delta] \geq 0 \\ &\Leftrightarrow \bar{c}^T \geq [0\ 0\ 4\theta\ 0\ \delta\theta] \\ &\Leftrightarrow [0\ 0\ \bar{c}_3\ 0\ \bar{c}_5] \geq [0\ 0\ 4\theta\ 0\ \delta\theta] \\ &\Leftrightarrow \begin{cases} \bar{c}_3 \geq 4\theta \\ \bar{c}_5 \geq \delta\theta \end{cases} \end{aligned}$$

Therefore, $\theta \leq \frac{\bar{c}_3}{4}$ and $\theta\delta \leq \bar{c}_5$. Depending on the sign of δ, we have

$$\theta \in [\frac{\bar{c}_5}{\delta}\,;\ \frac{\bar{c}_3}{4}] \text{ if } \delta < 0$$

$$\theta \in [-\infty\,;\ \frac{\bar{c}_3}{4}] \text{ if } \delta = 0$$

$$\theta \in [-\infty\,;\ \min\left\{\frac{\bar{c}_3}{4}, \frac{\bar{c}_5}{\delta}\right\}] \text{ if } \delta > 0$$