CO 370 Fall 2019: Homework 4

Due: December 3rd by 5:00pm

Instructions You will be graded not only on correctness, but also on clarity of exposition. You are allowed to talk with classmates about the assignment as long as (1) you acknowledge the people you collaborate with, (2) you write your solutions on your own, and (3) you are able to fully explain your solutions. In the models, always give a clear definition to your decision variables (in most cases, this means that you must explain what they represent in plain words). In case you run into trouble (a question is ambiguous, data provided have an issue, problem with the implementation, etc.), it is your responsibility to ask me or your TAs for clarifications in a timely manner.

Homework submission Your solutions are to be submitted on Crowdmark.

Question 1 [15 marks] Consider the following problem:

$$\begin{array}{rrcrcrcrcrcl} \max & 2x_1 & + & 4x_2 & + & 6x_3 & + & x_4 & + & 5x_5 & & \\ \text{s.t.} & 9x_1 & + & 13x_2 & + & 7x_3 & + & 14x_4 & + & 21x_5 & \leq & 40 \end{array} \tag{P}$$
$$x \in \{0,1\}^5.$$

Solve (P) using the branch-and-bound method. At any given node of the branch-and-bound tree, if the optimal LP solution is $\tilde{x}$ with $\tilde{x}_i \notin \mathbb{Z}$ and branching is required, always fully explore the subtree in which $x_i = 0$ before exploring the subtree in which $x_i = 1$. (In technical terms, perform a depth-first search on the branch-and-bound tree, always starting with the "$= 0$" branch.)

For every node, give the optimal LP solution and its objective function value (or write "infeasible"), and if branching is not required, specify why (pruning, infeasible, integer). Separately, draw the branch-and-bound tree.

Solution: We first reorder the variables such that $\frac{c_1}{a_1} \geq \cdots \geq \frac{c_5}{a_5}$:

$$\begin{array}{rrcrcrcrcrcl} \max & 6y_1 & + & 4y_2 & + & 5y_3 & + & 2y_4 & + & y_5 & & \\ \text{s.t.} & 7y_1 & + & 13y_2 & + & 21y_3 & + & 9y_4 & + & 14y_5 & \leq & 40 \end{array} \tag{P'}$$
$$x \in \{0,1\}^5.$$

where

$$x_3 = y_1, \quad x_2 = y_2, \quad x_5 = y_3, \quad x_1 = y_4, \quad x_4 = y_5.$$

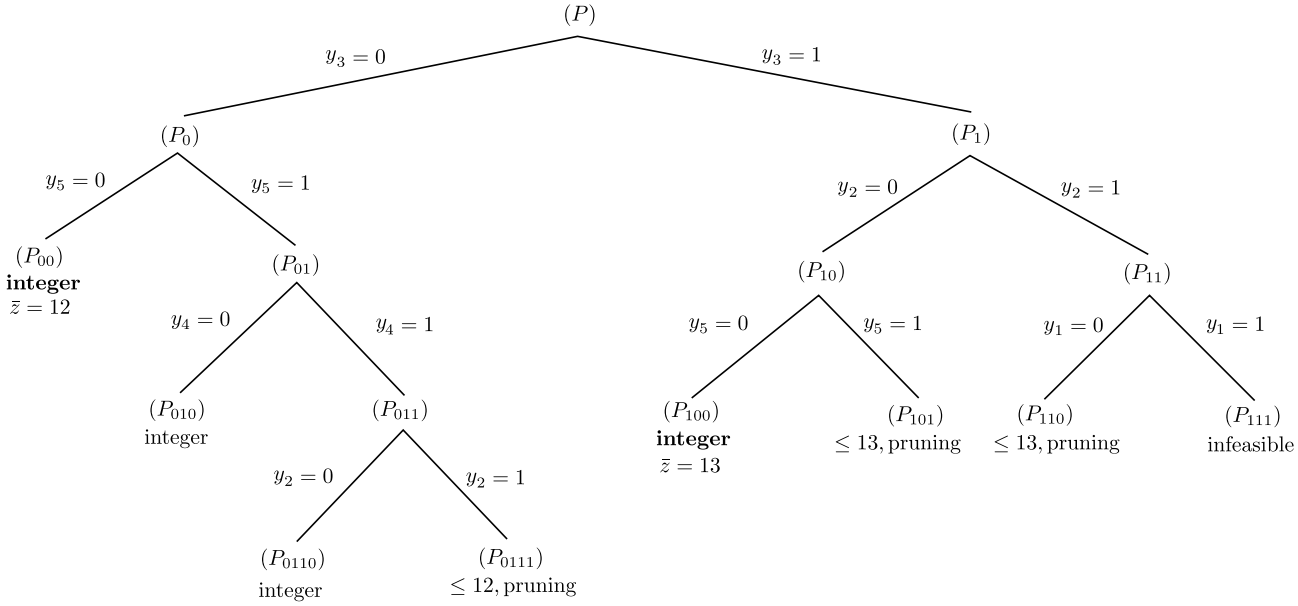
node	fixed	LP relaxation			best integer
(P')		$\tilde{y} = (1,\ 1,\ \frac{20}{21},\ 0,\ 0)$,	$\tilde{z} = 14 + \frac{16}{21}$		
(P'_0)	$y_3 = 0$	$\tilde{y} = (1,\ 1,\ \underline{0},\ 1,\ \frac{11}{14})$,	$\tilde{z} = 12 + \frac{11}{14}$		
(P'_{00})	$y_3 = 0, y_5 = 0$	$\tilde{y} = (1,\ 1,\ \underline{0},\ 1,\ \underline{0})$,	$\tilde{z} = 12$	**integer**	$\bar{z} = 12$
(P'_{01})	$y_3 = 0, y_5 = 1$	$\tilde{y} = (1,\ 1,\ \underline{0},\ \frac{2}{3},\ \underline{1})$,	$\tilde{z} = 12 + \frac{1}{3}$		$\bar{z} = 12$
(P'_{010})	$y_3 = 0, y_5 = 1, y_4 = 0$	$\tilde{y} = (1,\ 1,\ \underline{0},\ \underline{0},\ \underline{1})$,	$\tilde{z} = 11$	integer	$\bar{z} = 12$
(P'_{011})	$y_3 = 0, y_5 = 1, y_4 = 1$	$\tilde{y} = (1,\ \frac{10}{13},\ \underline{0},\ \underline{1},\ \underline{1})$,	$\tilde{z} = 12 + \frac{1}{13}$		$\bar{z} = 12$
(P'_{0110})	$y_3 = 0, y_5 = 1, y_4 = 1, y_2 = 0$	$\tilde{y} = (1,\ \underline{0},\ \underline{0},\ \underline{1},\ \underline{1})$,	$\tilde{z} = 9$	integer	$\bar{z} = 12$
(P'_{0111})	$y_3 = 0, y_5 = 1, y_4 = 1, y_2 = 1$	$\tilde{y} = (\frac{4}{7},\ \underline{1},\ \underline{0},\ \underline{1},\ \underline{1})$,	$\tilde{z} = 10 + \frac{3}{7}$	≤ 12, pruning	$\bar{z} = 12$
(P'_1)	$y_3 = 1$	$\tilde{y} = (1,\ \frac{12}{13},\ \underline{1},\ 0,\ 0)$,	$\tilde{z} = 14 + \frac{9}{13}$		$\bar{z} = 12$
(P'_{10})	$y_3 = 1, y_2 = 0$	$\tilde{y} = (1,\ \underline{0},\ \underline{1},\ 0,\ \frac{3}{14})$,	$\tilde{z} = 13 + \frac{3}{14}$		$\bar{z} = 12$
(P'_{100})	$y_3 = 1, y_2 = 0, y_5 = 0$	$\tilde{y} = (1,\ \underline{0},\ \underline{1},\ 1,\ \underline{0})$,	$\tilde{z} = 13$	**integer**	$\bar{z} = 13$
(P'_{101})	$y_3 = 1, y_2 = 0, y_5 = 1$	$\tilde{y} = (\frac{5}{7},\ \underline{0},\ \underline{1},\ 0,\ \underline{1})$,	$\tilde{z} = 10 + \frac{2}{7}$	≤ 13, pruning	$\bar{z} = 13$
(P'_{11})	$y_3 = 1, y_2 = 1$	$\tilde{y} = (\frac{6}{7},\ \underline{1},\ \underline{1},\ 0,\ 0)$,	$\tilde{z} = 14 + \frac{1}{7}$		$\bar{z} = 13$
(P'_{110})	$y_3 = 1, y_2 = 1, y_1 = 0$	$\tilde{y} = (\underline{0},\ \underline{1},\ \underline{1},\ \frac{2}{3},\ 0)$,	$\tilde{z} = 10 + \frac{1}{3}$	≤ 13, pruning	$\bar{z} = 13$
(P'_{111})	$y_3 = 1, y_2 = 1, y_1 = 1$	$\tilde{y} = (\underline{1},\ \underline{1},\ \underline{1},\ ?,\ ?)$		infeasible	$\bar{z} = 13$

An optimal integer solution is given by

$$y^* = [1 \quad 0 \quad 1 \quad 1 \quad 0]^T$$

and since $x_3 = y_1$, $x_2 = y_2$, $x_5 = y_3$, $x_1 = y_4$ and $x_4 = y_5$, we get

$$x^* = [1 \quad 0 \quad 1 \quad 0 \quad 1]^T \quad \text{with} \quad z^* = 13.$$



Question 2 [15 marks] Consider a matrix $A \in \mathbb{R}^{m \times n}$ such that:

- every element A_{ij} of A is either 0 or 1, and
- whenever $A_{ij} = 1$ and $A_{kj} = 1$ for some j and for some $i < k$, then we also have $A_{\ell j} = 1$ for all $i \leq \ell \leq k$.

Prove that A is totally unimodular.

Solution: In order to prove that A is TU, we need to prove that every square submatrix B of A is unimodular (i.e. $\det(B) \in \{-1, 0, 1\}$). Let $B \in \mathbb{R}^{p \times p}$ be any square submatrix of A. Observe that B possesses the same properties as A, i.e. (i) every element B_{ij} of B is either 0 or 1, and (ii) whenever $B_{ij} = 1$ and $B_{kj} = 1$ for some j and for some $i < k$, then we also have $B_{\ell j} = 1$ for all $i \leq \ell \leq k$.

We know that if a matrix B' is obtained from B by adding a multiple of one row to another, i.e.

$$B'_{ij} = \begin{cases} B_{ij} & \text{if } i \neq t \\ B_{tj} + \lambda B_{sj} & \text{if } i = t \end{cases}$$

for some $\lambda \in \mathbb{R}$ and $s \neq t$, then $\det(B') = \det(B)$. We will apply this operation multiple times: we subtract row 2 from row 1, then row 3 from row 2, then row 4 from row 3, ..., then row p from row $p-1$. Row p is left intact, which corresponds to subtracting a zero row from it. More rigorously, we construct a matrix $\tilde{B}$ that satisfies

$$\tilde{B}_{ij} = \begin{cases} B_{ij} - B_{i+1,j}, & \text{if } i < p \\ B_{pj}, & \text{if } i = p \end{cases} \quad = B^*_{ij} - B^*_{i+1,j},$$

where B^* is a matrix obtained by appending a row of zeros to B, i.e. $B^*_{ij} = B_{ij}$ for all $i, j \leq p$ and $B^*_{p+1,j} = 0$ for all j. Equivalently,

$$\tilde{B}_{ij} = \begin{cases} 0 & \text{if } B^*_{ij} = B^*_{i+1,j} & (a) \\ -1 & \text{if } B^*_{ij} = 0 \text{ and } B^*_{i+1,j} = 1 & (b) \\ +1 & \text{if } B^*_{ij} = 1 \text{ and } B^*_{i+1,j} = 0. & (c) \end{cases}$$

Observe that, by construction, B^* also has the same properties as A. Therefore, let us look at an individual column j of B^* (see Figure 1). If condition (b) is satisfied for some row h, i.e. if $B^*_{hj} = 0$ and $B^*_{h+1,j} = 1$, then condition (b) cannot be satisfied for any row $i \neq h$, as it would mean having two ones in the column that are separated by zeros. Similarly, if condition (c) is satisfied for some row h, i.e. if $B^*_{hj} = 1$ and $B^*_{h+1,j} = 0$, then condition (c) cannot be satisfied for any row $i \neq h$. As a consequence, we know that column j of $\tilde{B}$ has at most one element $+1$, at most one element -1, and all other elements are zeros. We have seen in class that this means that $\tilde{B}$ is totally unimodular, i.e. all submatrices of $\tilde{B}$ are unimodular. In particular, $\tilde{B}$ itself is unimodular, i.e. $\det(\tilde{B}) \in \{-1, 0, +1\}$. Since we constructed $\tilde{B}$ from B by applying operations that preserve the determinant, we have $\det(B) = \det(\tilde{B}) \in \{-1, 0, +1\}$.

$$B^*.e_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{B}.e_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad B^*.e_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{B}.e_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 1: Two example columns of B^* and $\tilde{B}$. Observe that the last entry of $B^*.e_j$ is always zero.