PART I: LINEAR OPTIMIZATION

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Linear Programming (LP) is the problem of maximizing or minimizing a **linear function** subject to a finite number of **linear constraints**.

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DEFINITION A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if $f(x) = \alpha^T x$ for some $\alpha \in \mathbb{R}^n$.

EXAMPLES

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x) = x_1 - x_2 + 2x_3$$

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x) = -5x_1 + \frac{1}{2}x_2$$

Definition A **linear constraint** is a constraint that takes one of the following three forms:

1. $g(x) \geq b$ or
2. $g(x) = b$ or
3. $g(x) \leq b$

where $g(x)$ is a linear function and $b \in \mathbb{R}$.

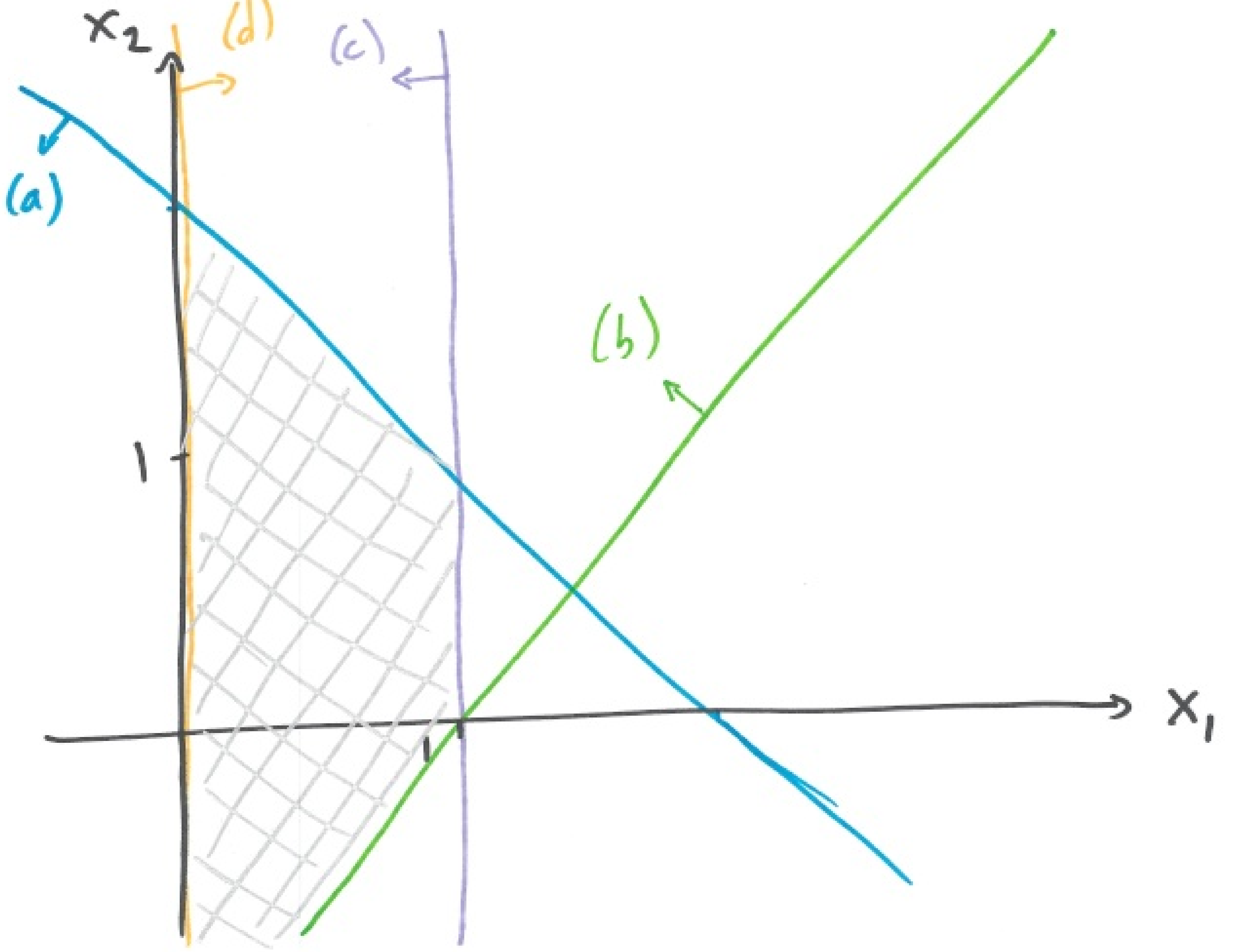
Examples

$$5x_1 + 0.3x_2 - x_3 = 5$$

$$12.7x_2 - x_5 \leq 4.3$$

A complete LP:

$$
\begin{array}{lrcll}
\max & 3x_1 - x_2 & & & \\
\text{s.t.} & x_1 + x_2 & \leq & 2 & \text{(a)} \\
& x_1 - x_2 & \leq & 1 & \text{(b)} \\
& x_1 & \leq & 1 & \text{(c)} \\
& x_1 & \geq & 0 & \text{(d)}
\end{array}
$$



x2
(d)
(c)
(a)
(b)
1
1
x1

Quizz

1) $\max \quad x_1$

$\text{s.t.} \quad x_2 + x_3 < 3$

$x_1 - x_4 \geq 1$

NOT AN LP

2) $\max \quad 3x_1 \cdot x_2$

$\text{s.t.} \quad x_1 \geq 3$

$x_1 + 3x_2 \leq 1$

NOT AN LP

3) $\min \quad x_1 + x_2$

$\text{s.t.} \quad x_1^2 + x_2^2 \leq 1$

NOT AN LP

4) $\min \quad x_1 + x_2$

IS AN LP (FINITE SET)

$\text{s.t.} \quad \alpha_1 x_1 + \alpha_2 x_2 \leq 1, \quad \forall (\alpha_1, \alpha_2) \in \{(1,0), (0,1), (1,1)\}$

<u>Note:</u> why not ">" ?

example: $\min \ x$
s.t. $x > 1$

undefined: $x = 1$ infeasible
$x = 1.00001$ not optimal

<u>Note:</u>

- $\min c^T x = -\max(-c^T x)$
- $g(x) \geq b \iff (-g)(x) \leq -b$
- $g(x) = b \iff \begin{cases} g(x) \geq b \\ g(x) \leq b \end{cases}$

A **Linear Programming (LP)** is characterized by

- decision variables,
- a linear objective function,
- a (finite) number of linear constraints.

Example 1

A company produces NH_3 and NH_4Cl.

component	stock
N	50 kmol
H	180 kmol
Cl	40 kmol

product	revenue
NH_3	\$40 / kmol
NH_4Cl	\$50 / kmol

Plan production to maximize profit.

Example 1

$$
\begin{array}{rrcrcr}
\max & 40x_1 & + & 50x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 50 \\
 & 3x_1 & + & 4x_2 & \leq & 180 \\
 & & & x_2 & \leq & 40 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Example 1

```
@variable(model, x1)
@variable(model, x2)

@objective(model, Max,    40 * x1 + 50 * x2          )

@constraint(model, N,          x1 +      x2 <= 50   )
@constraint(model, H,      3 * x1 +  4 * x2 <= 180  )
@constraint(model, Cl,                   x2 <= 40   )
@constraint(model, NH3,        x1           >= 0    )
@constraint(model, NH4Cl,                x2 >= 0    )
```

Example 1

```
julia> optimize!(model)

julia> objective_value(model)
2300.0

julia> value(x1)
20.000000000000004

julia> value(x2)
29.999999999999996
```

Tricks:

A. $\min \ x_3$

$\text{s.t.} \ |2x_1 - 3x_2 + 5x_3| \leq 8$

$\min \ x_3$

$\text{s.t.} \ y = 2x_1 - 3x_2 + 5x_3$

$-8 \leq y \leq 8$

B. $\min \ |2x_1 - 3x_2 + 5x_3|$

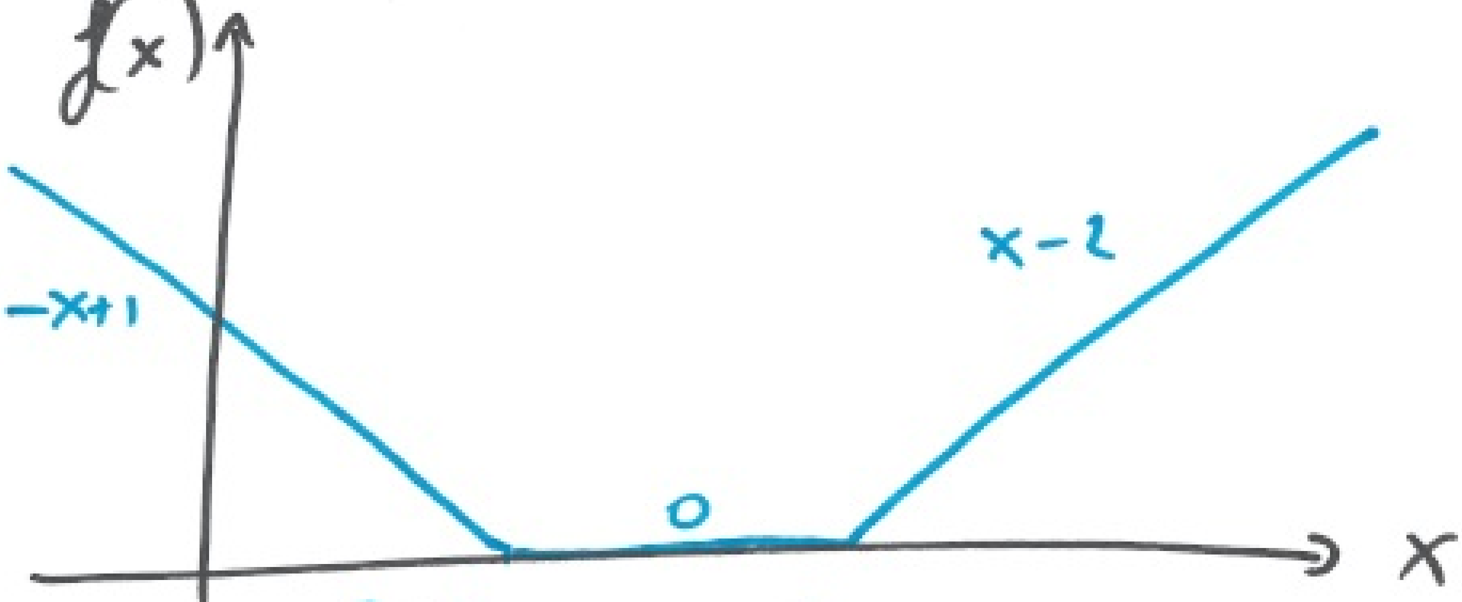
$\text{s.t} \ (\cdots)$

$\min \ y$

$\text{s.t.} \ -y \leq 2x_1 - 3x_2 + 5x_3 \leq y$

$(\cdots)$

C.



$$f(x) = \max\{-x+1,\ 0,\ x-2\}$$

$$\begin{aligned} \min \quad & (\ldots) \\ \text{s.t} \quad & \max\{-x+1,\ 0,\ x-2\} \leq 4 \end{aligned}$$

$$\begin{aligned} \min \quad & (\ldots) \\ \text{s.t.} \quad & -x+1 \leq 4 \\ & 0 \leq 4 \\ & x-2 \leq 4 \end{aligned}$$