REFRESHER: DIRECTED GRAPHS

DEF. A <u>directed graph</u> G is a pair (V, E), where

- V is a finite set (vertices)
- E is a set of ordered pairs (arcs) of elements of V

<u>EXAMPLE 1:</u>

$$V = \{s, a, b, t\}$$

$$E = \{sa, sb, ab, ba, at, bt\}$$

Notation:

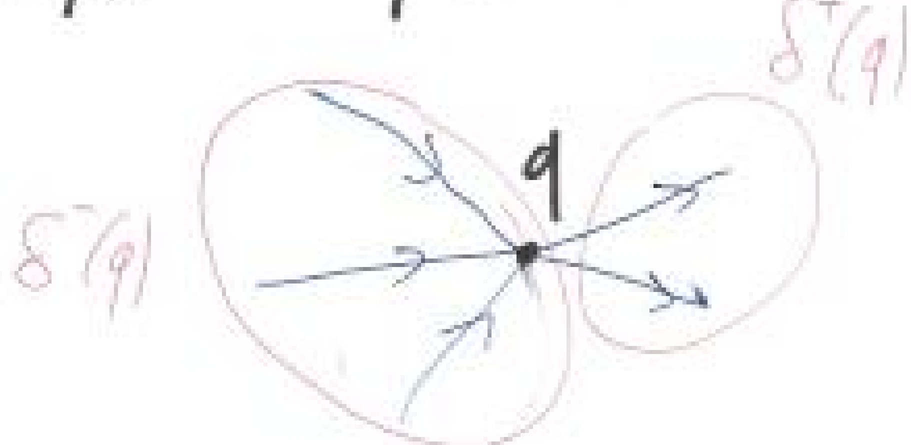
Let $e = uv \in E$ $\quad (u \xrightarrow{e} v)$

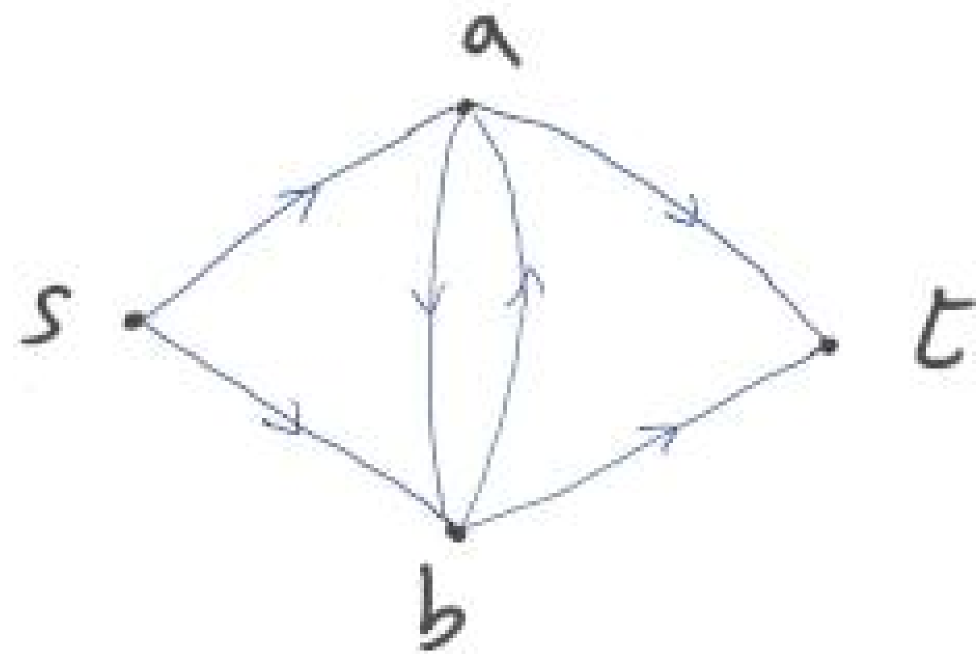
- e <u>leaves</u> u
- e <u>arrives at</u> v

For any $q \in V$

$$\delta^+(q) = \{ e \in E : e \text{ leaves } q \}$$

$$\delta^-(q) = \{ e \in E : e \text{ arrives at } q \}$$





$$\delta^+(a) = \{at, ab\}$$

$$\delta^-(a) = \{ba, sa\}$$

MAX S-T FLOW PROBLEM

Given ① $G = (V, E)$ directed graph

② "source" $s \in V$

③ "sink" $t \in V$, $t \neq s$

④ "capacity" u_e, for every $e \in E$.

- We want that for every $q \in V \setminus \{s, t\}$, the "flow" entering q = "flow" leaving q.
- The "flow" on $e \in E$ is at most u_e.
- Find maximum "flow" from s to t.

VAR: x_e : amount of flow on arc $e \in E$.

Notation: for $q \in V$,

$$f_x(q) = \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e$$

MAX S-T FLOW FORMULATION:

$$\max \quad f_x(s)$$

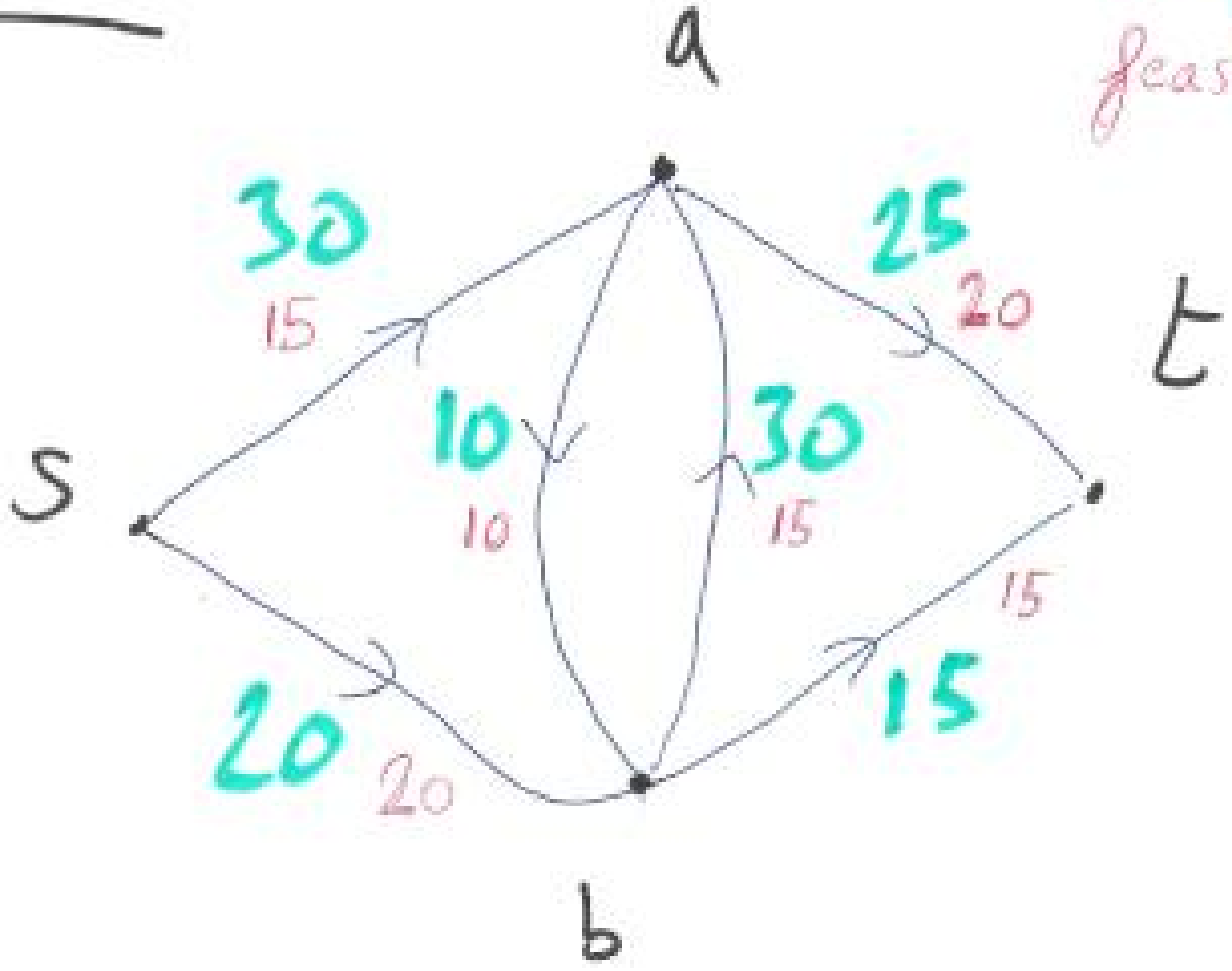
$$f_x(q) = 0 \quad \forall\, q \in V \setminus \{s,t\}$$

$$0 \leq x \leq u_e \quad \forall\, e \in E$$

EXAMPLE 1

Capacity u_e

feasible flow x_e



PROPERTY of max s-t flow problems

$$
\begin{aligned}
\max \quad & f_x(s) \\
\text{st.} \quad & f_x(q) = 0 \quad (q \neq s,t) \qquad (P) \\
& 0 \leq x_e \leq u_e
\end{aligned}
$$

If u_e is integer for all e, then among all the optimal solutions to (P), at least one is integral.

note: • if U_e is not integer:

$$s \xrightarrow[U_{st}=1.5]{} t$$

• there can still be other optimal solutions that are not integral.

Capacities
optimal solutio