PREVIOUS LECTURE

Max s-t flow formulation:

$$(P) \quad \begin{array}{ll} \max & f_x(s) \\ \text{s.t.} & f_x(q) = 0, \quad \forall q \in V \setminus \{s,t\} \\ & 0 \le x \le u \end{array} \qquad (u \ge 0)$$

where $f_x(q) = \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e$

PROPERTY: If $\underline{u \in \mathbb{Z}^n}$,

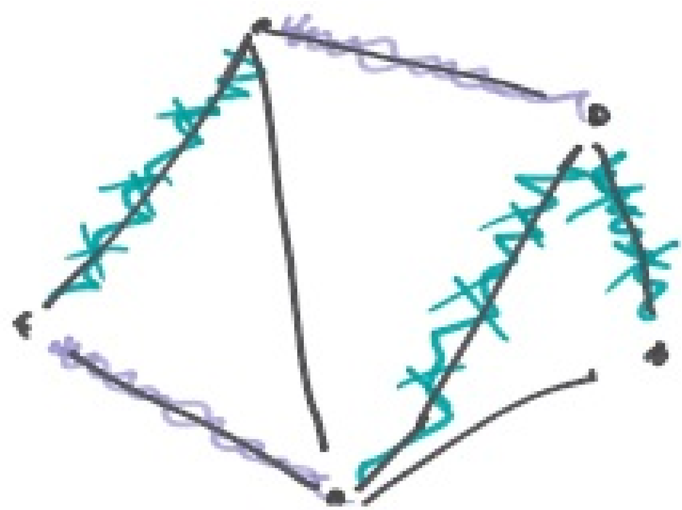
then $\exists\, x^*$ optimal for (P)

such that $\underline{x^* \in \mathbb{Z}^n}$.

APPLICATION OF PROPERTY: BIPARTITE MATCHINGS

Let $G = (V, E)$ be an (undirected) graph

DEF. $M \subseteq E$ is a matching if no two edges of M are incident to a same vertex



matching

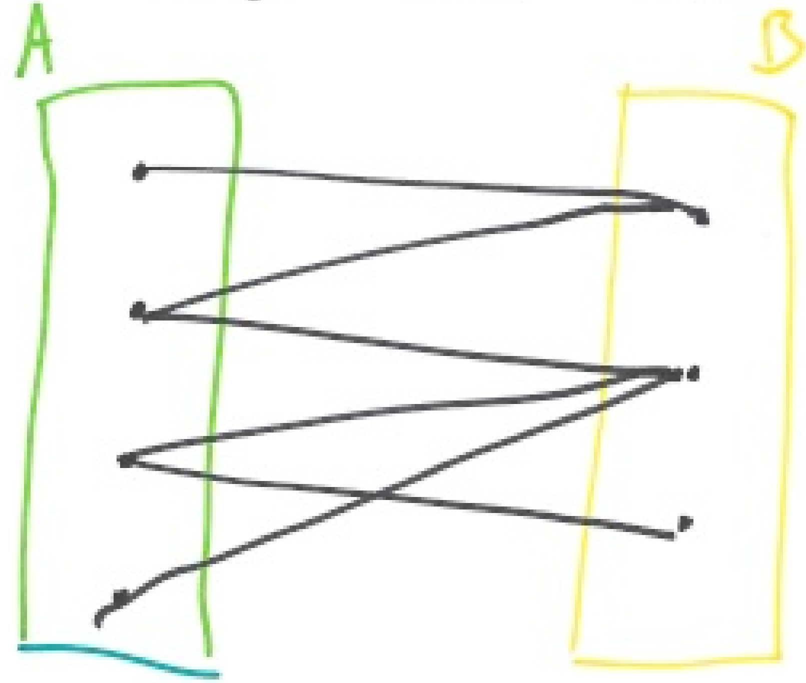
not a matching

DEF. G is <u>bipartite</u> if

$\exists$ a partition A, B of V, i.e.

- $A \cap B = \emptyset$
- $A \cup B = V$

such that every edge has one end in A and the other in B.



Problem Find a maximum-cardinality matching M in a bipartite graph G.

Naive formulation

VAR:

$$x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}, \quad \forall e \in E$$

MODEL:

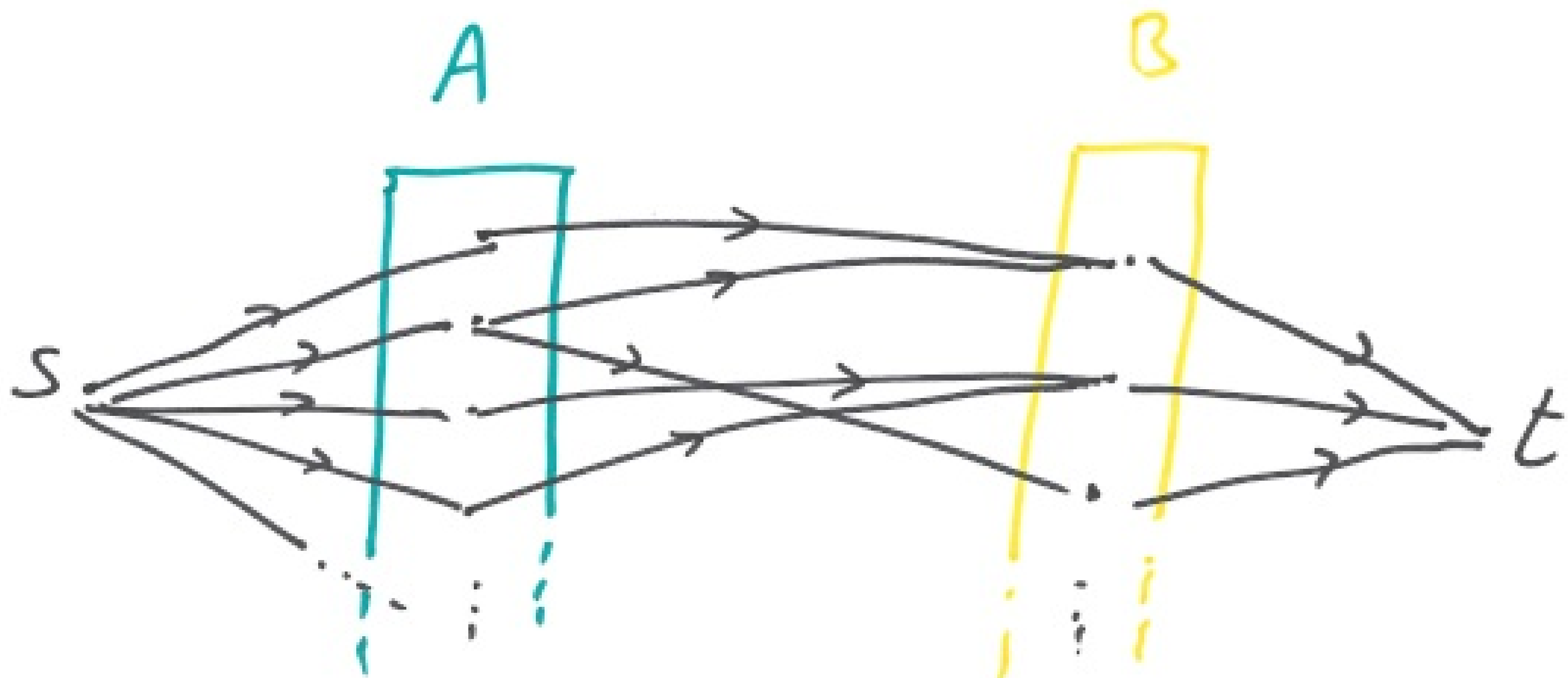
$$(Q) \qquad \begin{aligned} \max \ & \sum_{e \in E} x_e \\ \text{s.t.} \ & \sum_{e \in \delta(q)} x_e \leq 1, \quad \forall q \in V \\ & (x_e \in \{0, 1\}) \\ & 0 \leq x_e \leq 1, \quad \forall e \in E \\ & x_e \in \mathbb{Z}, \quad \forall e \in E \end{aligned}$$

Flow formulation

Assume $E = \{qr : q \in A, r \in B\}$

Let $G' = (V', E')$, where

- $V' = V \cup \{s, t\}$
- $E' = E \cup \{sq : \forall q \in A\} \cup \{qt : \forall q \in B\}$



$\underline{\text{VAR:}}$ x_e = flow on edge $e \in E'$

$\underline{\text{MAX-FLOW MODEL:}}$

(Q')

$$\max f_x(s)$$
$$\text{s.t. } f_x(q) = 0 \quad \forall q \in \underbrace{V' \setminus \{s,t\}}_{= V = A \cup B}$$
$$0 \le x_e \le 1 \quad \forall e \in E'$$

- $f_x(s)$ = flow from A to B $= \sum_{e \in E} x_e$
- $\forall q \in A$, $f_x(q) = \sum_{e \in \delta^+_E(q)} x_e - \sum_{e \in \delta^-_{E'}(q)} x_e$

$$= \sum_{e \in \delta_E(q)} x_e - x_{sq} = 0$$

$$x_{sq} \le 1 \implies \sum_{e \in \delta_E} x_e \le 1$$

So (Q') is equivalent to (Q) without $x_e \in \mathbb{Z}$

However, using the property of max-flow $\exists$ an optimal solution (a') where $x_e \in \mathbb{Z}_+$, $\forall e \in E$.

THE MIN-COST FLOW MODEL

DATA:

① $G = (V, E)$ directed graph

② capacity $u_e \geq 0$ $\forall e \in E$

③ unit cost $c_e \in \mathbb{R}$ $\forall e \in E$

④ net flow $b_q \in \mathbb{R}$ "pushed" (or "pulled" if $b_q < 0$) at q, $\forall q \in V$.

PROBLEM Minimize total cost

VAR: x_e = flow through arc $e \in E$

MODEL:

$$\min \sum_{e \in E} c_e x_e$$

(T)

$$\text{s.t.} \quad f_x(q) = b_q \qquad \forall q \in V$$

$$0 \leq x_e \leq u_e \qquad \forall e \in E$$

APPLICATION OF MIN-COST FLOW MODEL: TRANSSHIPMENT PROBLEM

Consider a road network.

The arcs are roads with

- given capacities, and
- given transportation costs.

The vertices q are either

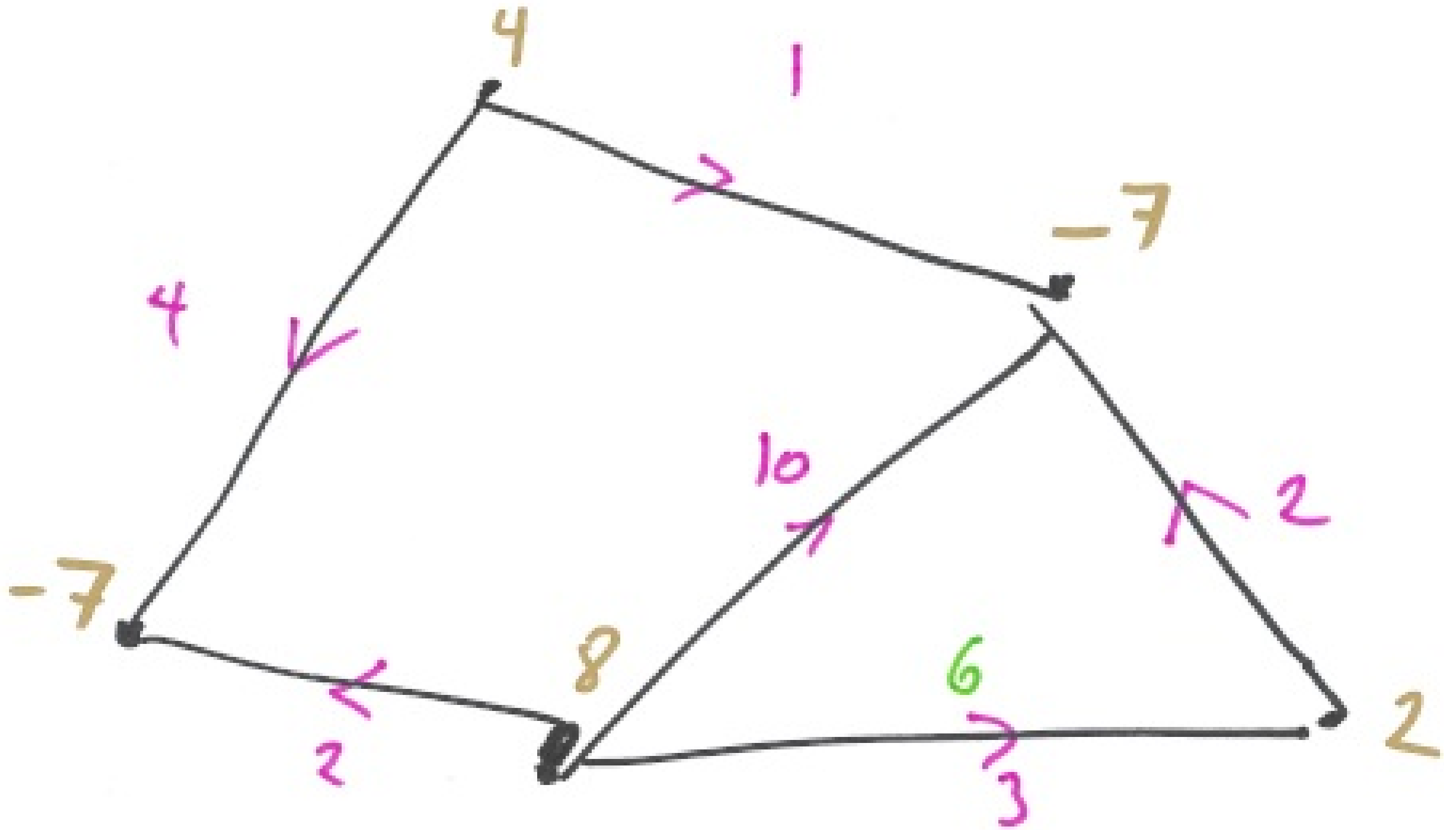
- storage centers that must send an amount $b_q > 0$ of goods, or
- retail centers that must receive an amount $b_q < 0$ of goods, or
- transshipment centers $(b_q = 0)$.

Minimize the cost of sending the goods.

b_q

c_e

u_e



PROPERTY: If $u \in \mathbb{Z}^n$ and $b \in \mathbb{Z}^m$, then $\exists\ x^*$ optimal for (T) such that $x^* \in \mathbb{Z}^n$.

APPLICATION OF PROPERTY:

SCHEDULING:

A bus of capacity p drives through cities $1, \dots, n$.

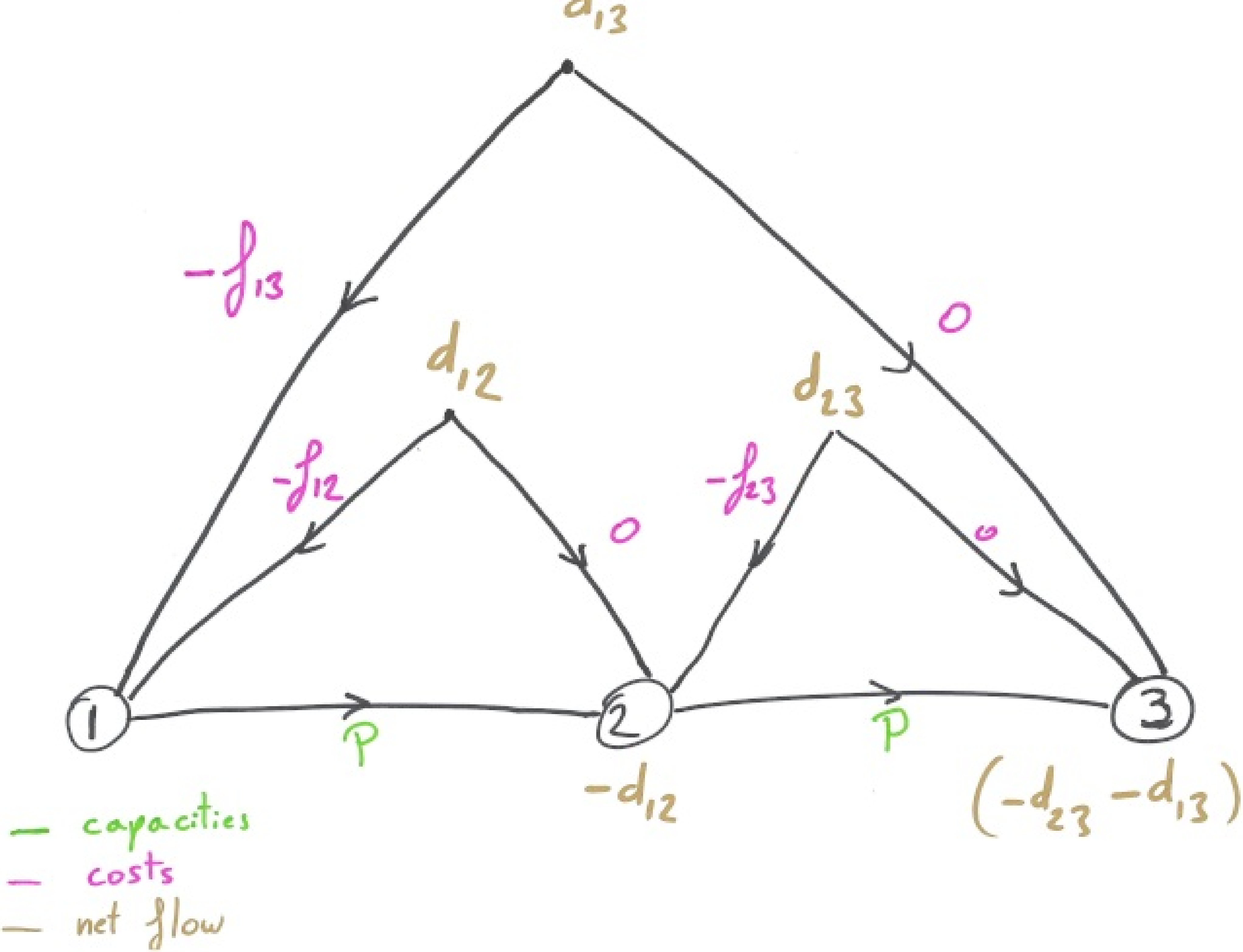
For every city $i < j$,

d_{ij} = demand to go from i to j

f_{ij} = fare to go from i to j.

Objective: decide how many passengers to carry for every $i \to j$ trip,

maximize fares collected.



d_{13}
$-f_{13}$
0
d_{12}
d_{23}
$-f_{12}$
0
$-f_{23}$
0
1
2
3
P
P
$-d_{12}$
$(-d_{23} - d_{13})$
— capacities
— costs
— net flow