IP tricks (part II)

3) We want $x \in \{a_1, \dots, a_m\}$
where $a_1, \dots, a_m \in \mathbb{R}$ are given constants.

Ex. $x \in \{3, 4.5, 10\}$

VAR:

$$x \in \mathbb{R}$$

$$y_i = \begin{cases} 1 & \text{if } x = a_i \\ 0 & \text{otherwise} \end{cases}$$

MODEL:

$$x = a_1 y_1 + a_2 y_2 + \cdots + a_m y_m$$

$$y_1 + y_2 + \cdots + y_m = 1$$

$$y_i \in \{0, 1\}, \quad \forall i = 1, \dots, m$$

4) Piecewise-linear function

DEF. A continuous function $f : [a_1, a_k] \to \mathbb{R}$ is piecewise-linear if there exist finitely many points $a_1 < a_2 < \cdots < a_k$ and corresp. function values $f_1, f_2, \dots, f_k \in \mathbb{R}$ such that:

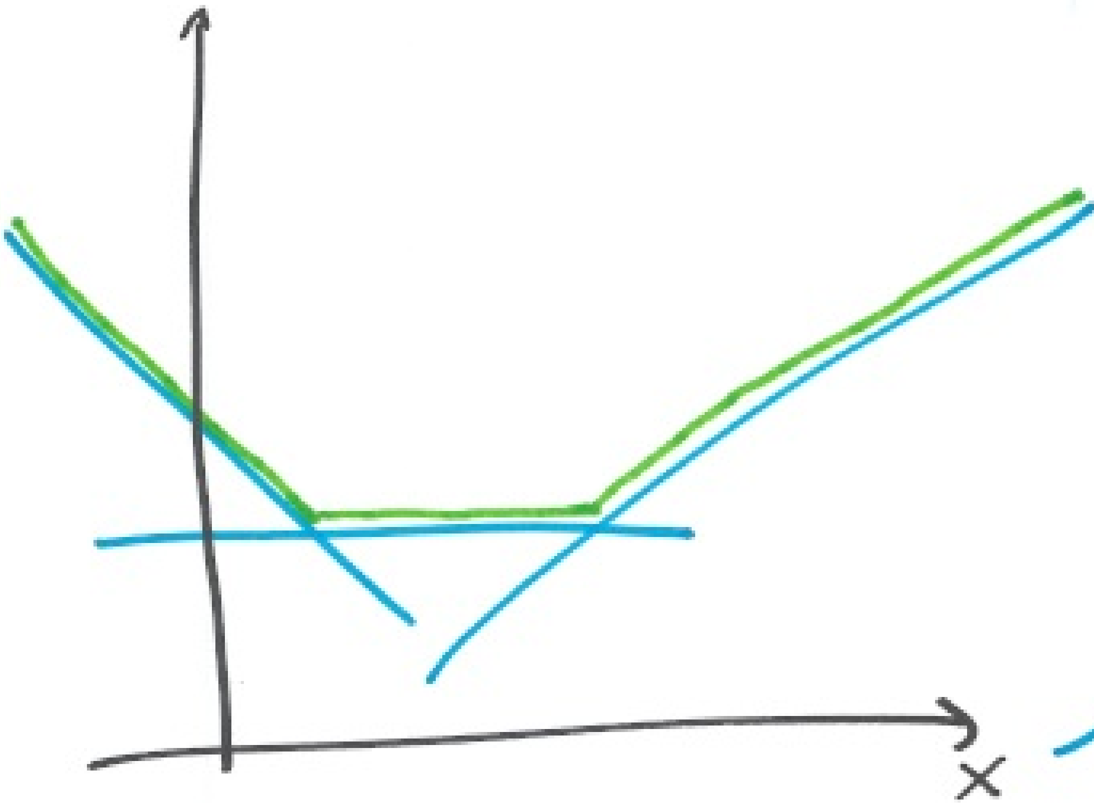
if $x \in [a_p, a_{p+1}]$,

i.e. if $x = \lambda a_p + (1-\lambda) a_{p+1}$ for some $0 \le \lambda \le 1$

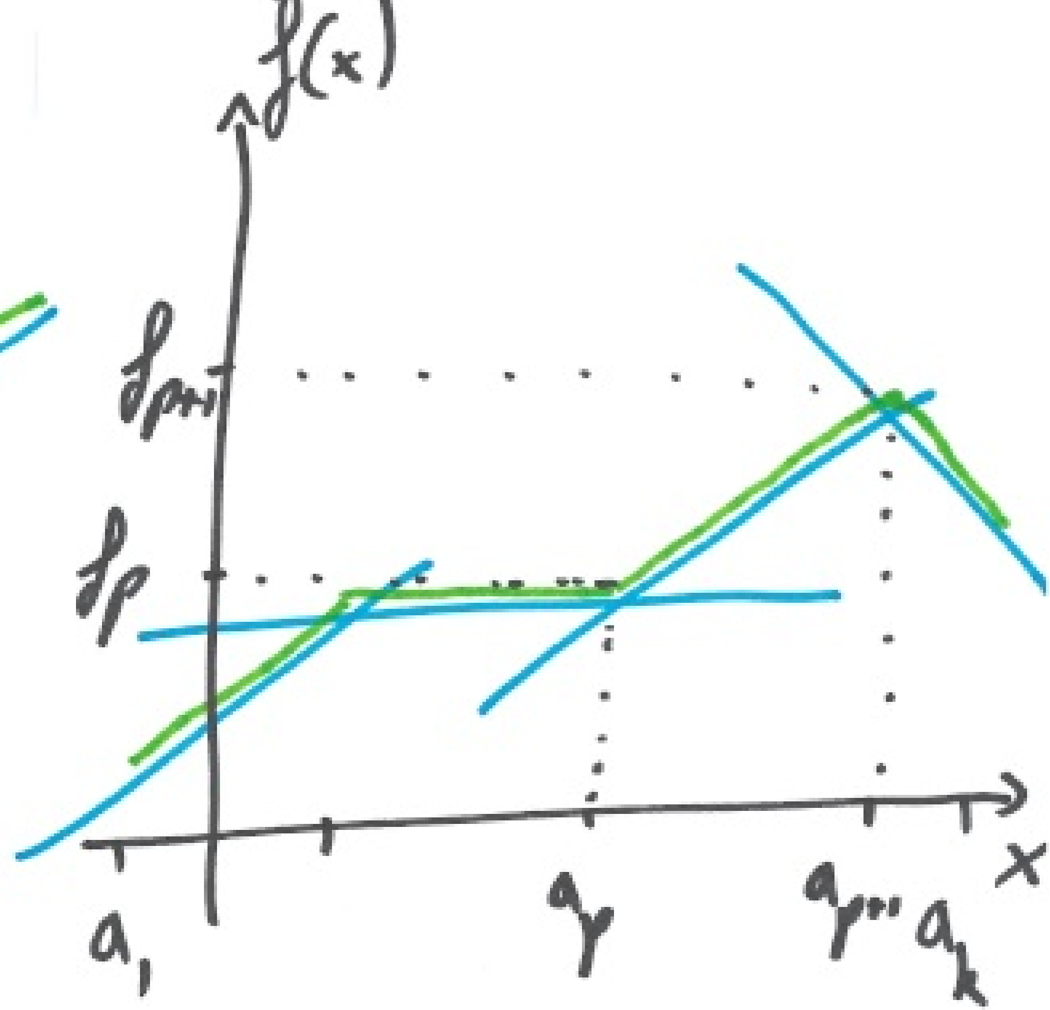
then $f(x) = \lambda f_p + (1-\lambda) f_{p+1}$.

In particular, $f(a_p) = f_p$ for $p = 1, \dots, k$.

$\max\{\dots,\dots,\dots\}$



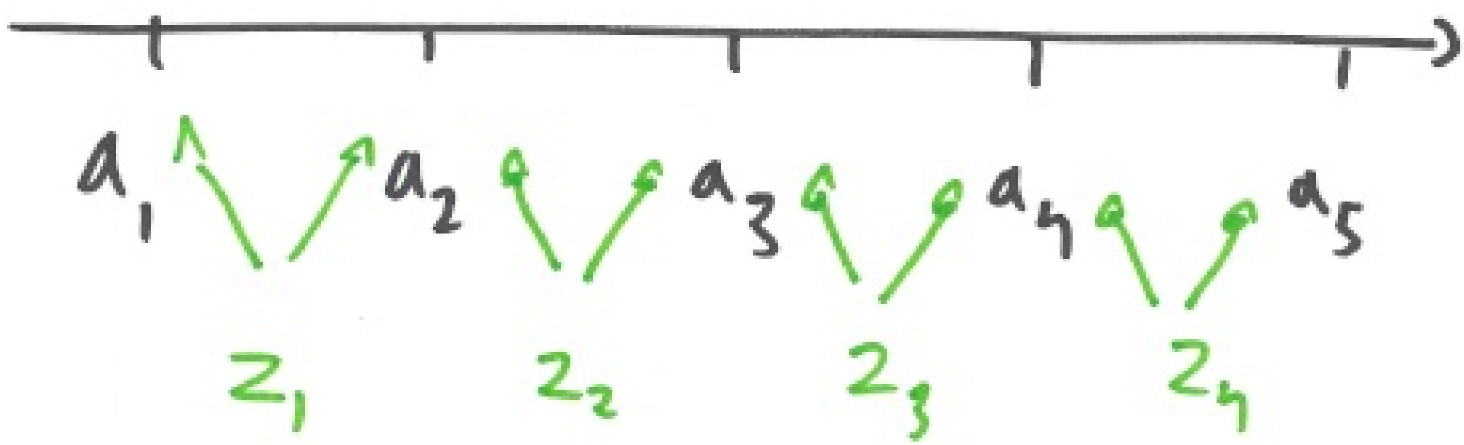
piecewise-linear



First let us rewrite the condition:

If $\begin{cases} x = \lambda_p a_p + \lambda_{p+1} a_{p+1} \\ \lambda_p , \ \lambda_{p+1} \geqslant 0 \\ \lambda_p + \lambda_{p+1} = 1 \end{cases}$

then $f(x) = f_p \lambda_p + f_{p+1} \lambda_{p+1}$



a_1
a_2
a_3
a_4
a_5
z_1
z_2
z_3
z_4

VAR:

$$x \in [a_1, a_k]$$

$$y = f(x)$$

λ_p : multiplier for a_p

$$z_p = \begin{cases} 1 & \text{if } \lambda_p, \lambda_{p+1} \text{ are allowed to be } > 0 \\ 0 & \text{otherwise} \end{cases}$$

MODEL:

$$
\begin{aligned}
x &= \lambda_1 a_1 + \cdots + \lambda_k a_k \\
y &= \lambda_1 f_1 + \cdots + \lambda_k f_k \\
& \lambda_1 + \cdots + \lambda_k = 1 \\
& \lambda_1, \ldots, \lambda_k \geq 0 \\
& z_1, \ldots, z_{k-1} \in \{0, 1\} \\
& z_1 + \cdots + z_{k-1} = 1
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1 &\leq z_1 \\
\lambda_2 &\leq z_1 + z_2 \\
\lambda_3 &\leq z_2 + z_3 \\
\lambda_4 &\leq z_3 + z_4 \\
&\vdots \qquad \ddots \\
\lambda_{k-1} &\leq z_{k-2} + z_{k-1} \\
\lambda_k &\leq z_{k-1}
\end{aligned}
$$

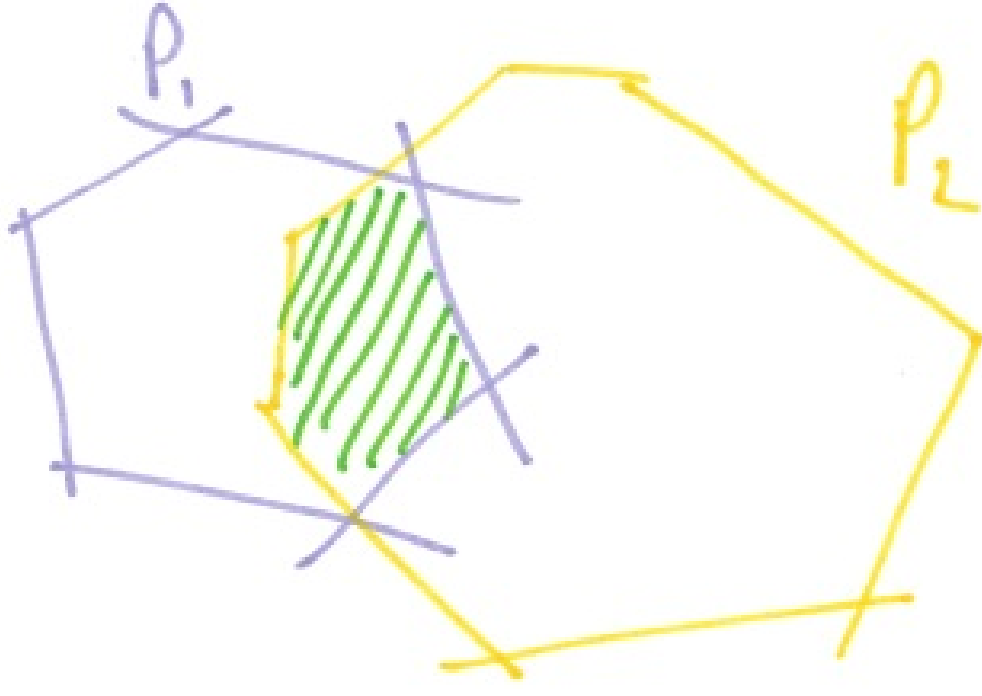
5) Optimizing over intersection of polyhedra

Let $P_1, P_2 \subseteq \mathbb{R}^n$:

$$P_1 = \{ x \in \mathbb{R}^n : A^1 x \leq b^1 \}$$

$$P_2 = \{ x \in \mathbb{R}^n : A^2 x \leq b^2 \}$$

Ex.



MODEL:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A^1 x \le b^1 \\ & A^2 x \le b^2 \end{aligned}$$

no need for tricks here!

6) Optimizing over union of polyhedra

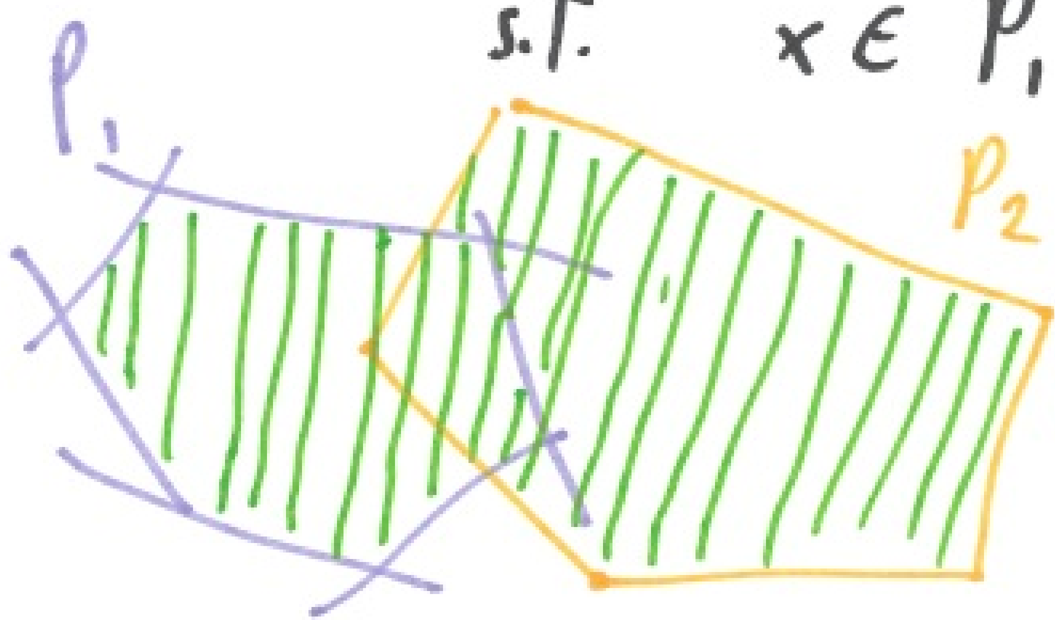
Let $P_1, P_2 \subseteq \mathbb{R}^n$

$$P_1 = \{x \in \mathbb{R}^n : A^1 x \leq b^1\}$$

$$P_2 = \{x \in \mathbb{R}^n : A^2 x \leq b^2\}$$

We want
$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in P_1 \cup P_2 \end{aligned}$$

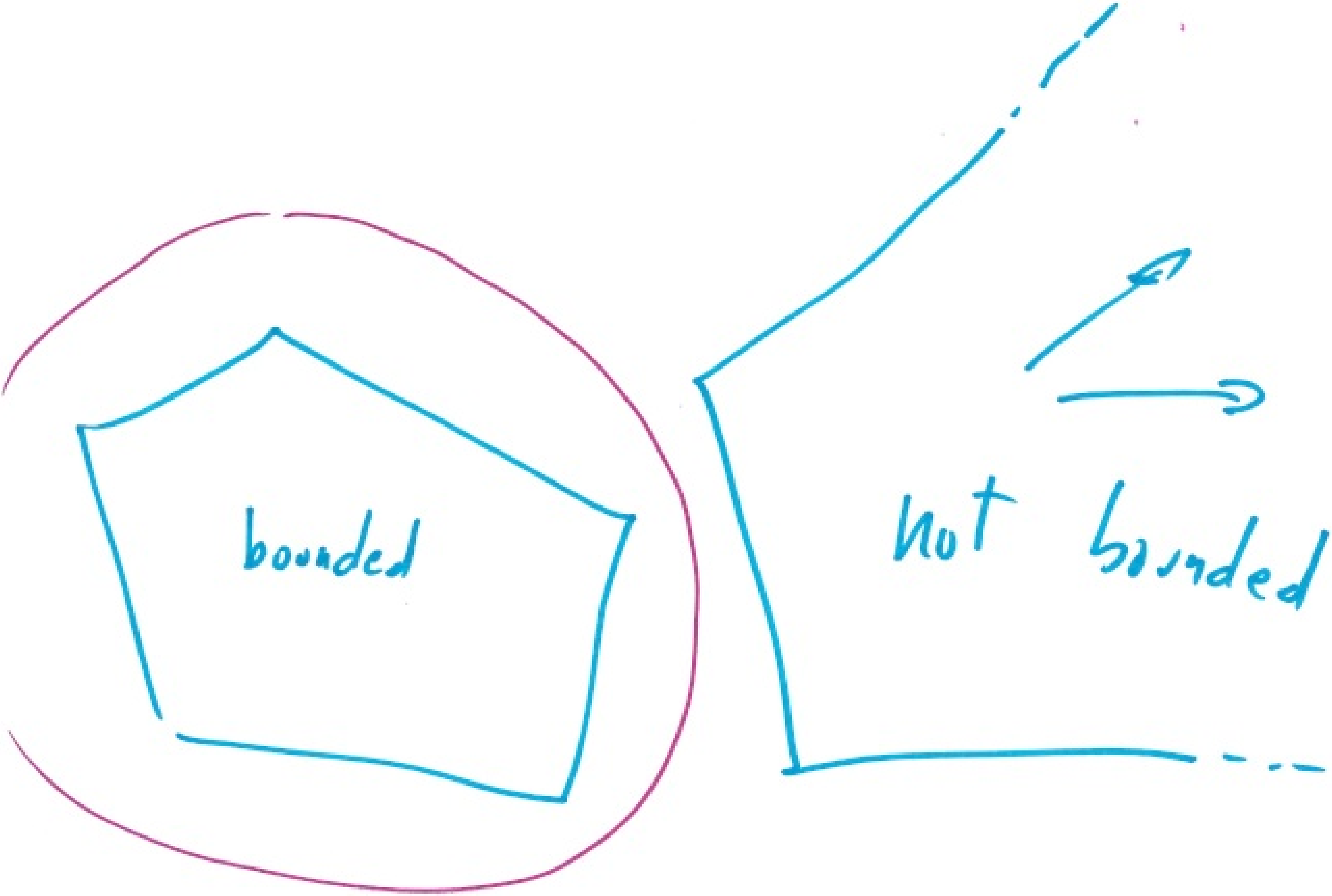
Ex:



We will tackle the special case where P_1 and P_2 are bounded.

We assume wlog $0 \leq x \leq U \ , \ \forall x \in P_1$

$0 \leq x < U \ , \ \forall x \in P_2$


bounded
not bounded

VAR

$$x \in P_1 \cup P_2$$

$$x^i \begin{cases} \in P_i & \text{if we "take" } x \text{ from } P_i \\ = 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if we "take" } x \text{ from } P_i \\ 0 & \text{otherwise} \end{cases}$$

MODEL

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & y_1, y_2 \in \{0, 1\} \\
& y_1 + y_2 = 1 \\
& x = x^1 + x^2 \\
& 0 \le x^1 \le y_1 U \\
& 0 \le x^2 \le y_2 U \\
& A^1 x^1 \le y_1 b^1 \\
& A^2 x^2 \le y_2 b^2
\end{aligned}
$$