PART III: SOLVING LPs

A) Standard equality form (SEF)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$c \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$$

Obtaining SEF:

- Inequality: $a^T x \le b \rightarrow a^T x + s = b$
  $s \ge 0$

- Free variable: $x_j \in \mathbb{R} \rightarrow$
  <u>substitute</u> $x_j = y_1 - y_2$
  $y_1, y_2 \ge 0$

B) find a solution to:

$$Ax = b \quad \text{where } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m}$$

Case 1: $\text{rank}(A) < m$

- rows of A are lin. dep.
- lin. dep. rows can be eliminated

Ex 1.

$$\begin{cases} 2x_1 + 3x_2 - x_3 + x_4 = 1 & (a) \\ x_1 + x_3 = 2 & (b) \\ 3x_1 + 3x_2 + x_4 = 3 & (c) \end{cases}$$

rows are lin. dep:

(a) + (b) = (c)

$\Rightarrow$ (c) is redundant

Ex 2.

$$\begin{cases} 3x_1 + 2x_2 = 2 & (d) \\ -x_1 - x_2 = 1 & (e) \\ x_1 = 0 & (f) \end{cases}$$

$(d) + 2 \cdot (e) \Rightarrow \quad x_1 = 4$

$x_1 = 0$ and $x_1 = 4 \Rightarrow$ INFEASIBLE

Case 2: $\operatorname{rank}(A) = m$, A is square $(m = n)$

$\Rightarrow A$ invertible

$$Ax = b \iff x = A^{-1}b$$

unique solution

Case 3: rank(A) = m, m < n

Ex. 3

$$\begin{cases} \boxed{2}\, x_1 + \boxed{3}\, x_2 - x_3 + x_4 = 1 \\ \boxed{1}\, x_1 + \boxed{} \quad + x_3 = 2 \end{cases}$$

- find a square $m \times m$ <u>invertible</u> submatrix B of A, for example:

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

- Set variables not involved in B to zero,

  for example: $x_3 = 0$, $x_4 = 0$

The problem becomes

$$\begin{cases} 2x_1 + 3x_2 = 1 \\ 1x_1 = 2 \end{cases}$$

$\rightarrow$ Use case 2:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\rightarrow$ $x = (2, -1, 0, 0)$

c) solve :

$$\min \; c^T x \qquad \text{(P)}$$
$$A x = b$$
$$x \geq 0$$

DEF. A column subset $\mathcal{B} \subseteq \{1, \dots, n\}$ is a basis of (P) if :

- $|\mathcal{B}| = m$
- $B = A_{\mathcal{B}}$ is invertible.

DEF

- $j \in \mathcal{B}$ is a basic column
- $\mathcal{N} = \{1, \dots, n\} \setminus \mathcal{B}$
- $j \in \mathcal{N}$ is a nonbasic column
- $B = A_{\mathcal{B}}$ is a basis matrix
- $N = A_{\mathcal{N}}$
- for simplicity, we write

$$A = [\, B \mid N \,]$$

$$c^T = [\, c_{\mathcal{B}}^T \mid c_{\mathcal{N}}^T \,]$$

ignoring column order.

(abuse of notation)

- if $\bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}$ satisfies:

$$\begin{cases} \bar{x}_N = 0 \\ \bar{x}_B = B^{-1} b \end{cases}$$

then $\bar{x}$ is the <u>basic solution</u> corresponding to B.

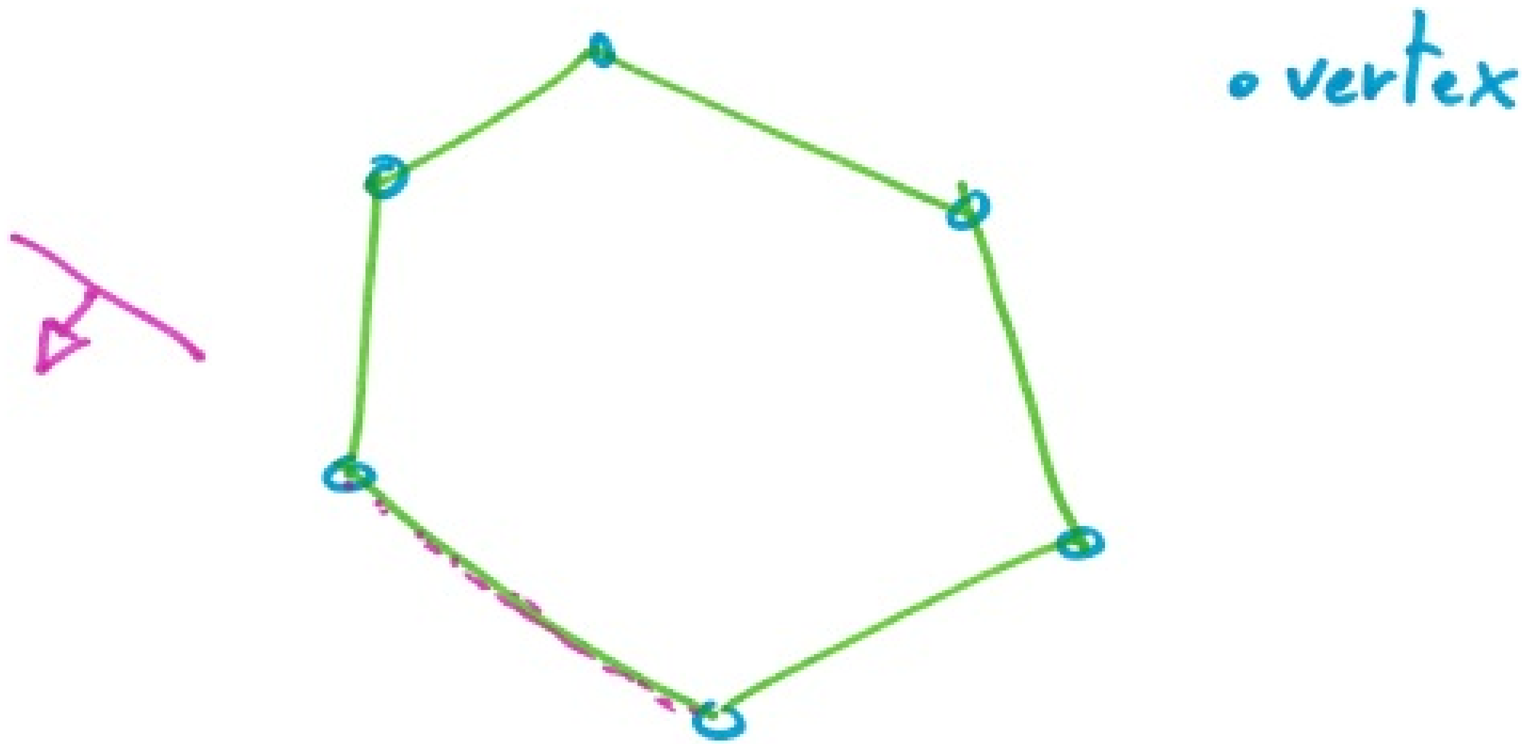
- if $\bar{x} \geqslant 0$, then $\bar{x}$ is a <u>basic feasible solution</u>.

Theorem: If

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

has an optimal solution, then it has an <u>optimal basic feasible solution.</u>

THEOREM: Basic feasible solutions are the vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$.



$\Rightarrow$ naive algorithm for LP:

- enumerate all m-column subsets
- if basis, compute corresp. basic solution
- if feasible, compute objective value
- keep best basic feasible solution

Observations:

O1) consider

$$z_1^* = \min \; c^T x \quad \text{s.t. } Ax = b, \; x \geq 0 \qquad \text{(LP1), and}$$

$$z_2^* = \min \; c^T x + C \quad \text{s.t. } Ax = b, \; x \geq 0 \qquad \text{(LP2)}$$

$z_1^* \neq z_2^*$, but

- $z_2^* = z_1^* + C$
- (LP1) and (LP2) have the same optimal solutions x^*.

02) $\forall G \in \mathbb{R}^{q \times m}$,

$$Ax = b \implies GAx = Gb$$

Ex. 4:

Let $G \in \mathbb{R}^{1 \times 2}$, $G = [1 \quad 1]$

$$\begin{cases} 2x_1 + 3x_2 - x_3 + x_4 = 1 \\ 1x_1 \qquad\quad + x_3 \qquad = 2 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$G.A = [1 \quad 1] \cdot \begin{bmatrix} & A & \end{bmatrix} = [3 \quad 3 \quad 0 \quad 1]$$

$$G.b = [1 \quad 1] \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3$$

$$\implies 3x_1 + 3x_2 \qquad + x_4 = 3$$

03) $\forall\, G \in \mathbb{R}^{m\times m}$ <u>invertible</u>,

$$Ax = b \iff GAx = Gb$$

<u>proof</u>:

$$Ax = b \overset{02}{\Rightarrow} GAx = Gb \overset{02}{\Rightarrow} G^{-1}GAx = G^{-1}Gb$$

$$\Rightarrow Ax = b$$

THEOREM Consider

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

Let $\mathcal{B}$ be a basis of (P), and let $B = A_{\mathcal{B}}$.

Then

$$c_{\mathcal{B}}^T B^{-1} b + \begin{aligned} \min\ & (c^T - c_{\mathcal{B}}^T B^{-1} A) x \\ \text{s.t. } & B^{-1} A x = B^{-1} b \\ & x \geq 0 \end{aligned}$$

is equivalent to (P).

proof

- By O3, $B^{-1}Ax = B^{-1}b \iff Ax = b$
- $(c^T - c_B^T B^{-1}A)x$

$$= c^T x - c_B^T B^{-1} Ax$$

$$= c^T x - c_B^T B^{-1} b \quad \text{constant}$$

By O1, this corresponds to the objective of (P).

Def. Given a basis $\mathcal{B}$, we define
corresponding:

- $\bar{c}^T = c^T - c_{\mathcal{B}}^T B^{-1} A$, the reduced cos
- $\bar{A} = B^{-1} A$
- $\bar{b} = B^{-1} b$
- simplex tableau:

$$\begin{aligned} \min \quad & \bar{c}^T x \\ & \bar{A} x = \bar{b} \\ & x \geq 0 \end{aligned}$$

Properties of a simplex tableau:

Theorem: Let $\bar{A} = [\,\bar{B} \mid \bar{N}\,]$, and
$\bar{c}^T = [\,\bar{c}_B^T \mid \bar{c}_N^T\,]$.

(1) $\bar{B} = I$

(2) $\bar{c}_B = 0$

<u>proof</u>: (1) $\bar{A} = [\,\bar{B} \mid \bar{N}\,] = B^{-1}A$

$$= B^{-1}[\,B \mid N\,]$$

$$= [\,B^{-1}B \mid B^{-1}N\,]$$

$$= [\,I \mid B^{-1}N\,]$$

(2) $[\,\bar{c}_B \mid \bar{c}_N\,] = [\,c_B^T \mid c_N^T\,] - c_B^T[\,I \mid B^{-1}$

$$= [\,c_B^T \mid c_N^T\,] - [\,c_B^T \mid c_B^T B^{-1}$$

$$= [\,c_B^T - c_B^T \mid c_N^T - c_B^T B^{-1}N\,]$$

$$= [\,0 \mid c_N^T - c_B^T B^{-1}N\,]$$