Previous lecture

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A x = b \\ & x \geq 0. \end{aligned} \qquad (P)$$

Given a basis $\mathcal{B}$, (P) is equivalent to

$$\begin{aligned} \min \quad & \bar{c}^T x \\ \text{s.t.} \quad & \bar{A} x = \bar{b} \\ & x \geq 0, \end{aligned} \qquad \text{(tableau)}$$

where

$$B = A_{\mathcal{B}}$$

$$\bar{c}^T = c^T - c_{\mathcal{B}}^T B^{-1} A$$

$$\bar{A} = B^{-1} A$$

$$\bar{b} = B^{-1} b$$

properties of tableau:

$$\bar{c}^T = [\ 0^T \mid \bar{c}_{\mathcal{N}}^T\ ]$$

$$\bar{A} = [\ \underbrace{I}_{\mathcal{B}} \mid \bar{A}_{\mathcal{N}}\ ]$$

corresponding basic solution:

$$x_{\mathcal{N}} = 0 \quad \text{so} \quad \bar{A}x = [\ I \mid \bar{A}_{\mathcal{N}}\ ]\begin{bmatrix} x_{\mathcal{B}} \\ \hline x_{\mathcal{N}} \end{bmatrix} = \bar{b}$$

$\leftarrow 0$

$$\text{yields} \quad I x_{\mathcal{B}} = \bar{b}$$

i.e., $x = \begin{bmatrix} \bar{b} \\ \hline 0 \end{bmatrix} \big\} \mathcal{B}$

note: basic feasible solutions $\Leftrightarrow$ vertices of (P)

We want to go from a feasible basis to a better feasible basis (without enumerating all bases)

Def. Let $\ell \in \mathcal{B}$, $e \in \mathcal{N}$.

A pivot is the action of creating

$$\mathcal{B}' = \mathcal{B} \setminus \{\ell\} \cup \{e\}.$$

x_e is the entering variable.

x_ℓ is the leaving variable.

If we pivot away from current basis,

- one nonbasic variable, x_e (currently $= 0$) will become basic (hence ≥ 0).
- the value of the basic variables will change to preserve feasibility (≥ 0)
- how will the objective value change?
  - $\forall j \in \mathcal{B}$, $\bar{c}_j = 0$ so x_j has no impact
  - $\forall j \in \mathcal{N} \setminus \{e\}$, $\bar{x}_j = 0$ and stays zero $\rightarrow$ no impact
  - x_e can increase, so objective can decrease if $\bar{c}_e < 0$

Example tableau

$$
\begin{array}{lllllllll}
\min & & & & 3x_4 & -2x_5 & & \\
& x_1 & & & +2x_4 & -x_5 & = & 2 \\
& & x_2 & & -x_4 & +x_5 & = & 1 \\
& & & x_3 & & +2x_5 & = & 1 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0
\end{array}
$$

basic solution : $\bar{x} = (2, 1, 1, 0, 0)$

In the example,

x_5 enters the basis $\Rightarrow$

(all other nonbasics, i.e. x_4, are fixed to 0)

$$x_1 = 2 + x_5 \geq 0$$
$$x_2 = 1 - x_5 \geq 0 \rightarrow x_5 \leq 1$$
$$x_3 = 1 - 2x_5 \geq 0 \rightarrow 2x_5 \leq 1$$

$\Rightarrow$ x_5 will increase to $\frac{1}{2}$

x_3 become zero, can <u>leave</u> the basis

When x_e increases:

if $\bar{a}_{ie} \leq 0$, x_j increases or stays unchanged

if $\bar{a}_{ie} > 0$, x_j decreases:

when x_e reaches $\frac{\bar{b}_i}{\bar{a}_{ie}}$, x_j reaches 0

$\Rightarrow$ Ratio test:

by how much can x_e increase?

$$\lambda = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ie}} \;\middle|\; \bar{a}_{ie} > 0 \right\}$$

Leaving variable: any i-th basic variable such that $\bar{a}_{ie} > 0$ and $\lambda = \frac{\bar{b}_i}{\bar{a}_{ie}}$.

More formally,

Let x_e enter the basis (with $\bar{c}_e < 0$).
Consider x_j where $j \in \mathcal{B}$, j is the i-th basic variable.

$$x_j \quad \overbrace{+ \cdots +}^{0} \quad \bar{a}_{ie} x_e = \bar{b}_i$$

So $x_j = \bar{b}_i - \bar{a}_{ie} \cdot x_e$

Simplex method

Given a basis $\mathcal{B}$ such that $\bar{b} \geq 0$,

- choose entering variable x_e : $\bar{c}_e < 0$
- choose pivot row

$$i = \underset{i}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{\bar{a}_{ie}} \;\middle|\; \bar{a}_{ie} > 0 \right\}$$

- leaving variable x_ℓ is the basic variable in row i.
- pivot (x_e enters, x_ℓ leaves)

The algorithm ends when

- there is no entering variable (optimality)
- there is no leaving variable (unboundedness)

DUALITY

Consider $z = \max \; 2x_1 + x_2$

$$\begin{aligned} \text{s.t.} \quad x_1 + 2x_2 &\leq 2 \\ x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 0.5 \end{aligned} \qquad (P)$$

Consider $\bar{x} = (1, 0.5)$, $\bar{z} = 2.5$

Q: Is $\bar{x}$ optimal?

A: It is optimal if $2x_1 + x_2 \leq 2.5$ $\forall x$ feasible

Q: Could we show $2x_1 + x_2 \leq U$ for some U

A: take linear combinations of constraints (with ≥ 0 coefficients)

$\downarrow$

$$
\begin{array}{rcl}
x_1 + 2x_2 \leq 2 & & (\times\ 1/3) \\
x_1 + x_2 \leq 2 & & (\times\ 1) \\
x_1 - x_2 \leq 0.5 & & (\times\ 2/3) \\
\hline
2x_1 + x_2 \leq 3 & &
\end{array}
$$

(still not sure $\bar{x}$ is optimal)

General approach:

$$x_1 + 2x_2 \leq 2 \quad (\times\ y_1)$$
$$x_1 + x_2 \leq 2 \quad (\times\ y_2)$$
$$x_1 - x_2 \leq 0.5 \quad (\times\ y_3)$$

$$(y_1 + y_2 + y_3)\, x_1 + (2y_1 + y_2 - y_3)\, x_2 \leq 2y_1 + 2y_2 + 0.5y_3$$

we want: $2x_1 + x_2 \leq 2y_1 + 2y_2 + 0.5\,y_3$

It will work as long as:

$$y_1 + y_2 + y_3 = 2$$
$$2y_1 + y_2 - y_3 = 1$$
$$y_1,\ y_2,\ y_3 \geq 0$$

in which case it gives $U = 2y_1 + 2y_2 + 0.5\,y_3$

We want the strongest possible upper bound U.

$$U = \min \; 2y_1 + 2y_2 + 0.5\,y_3$$

$$\begin{aligned} y_1 + y_2 + y_3 &= 2 \\ 2y_1 + y_2 - y_3 &= 1 \\ y_1, \; y_2, \; y_3 &\geq 0 \end{aligned} \qquad (D)$$

(D) is the <u>dual</u> of (P)

<u>Note</u>: $y^* = (1, 0, 1)$ feasible for (D)

gives $U = 2.5 \Rightarrow \bar{x}$ optimal for (P)