Theorem (weak duality)

Consider

$$(P)\quad \begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b, \\ & x \text{ free} \end{aligned} \qquad , \qquad (D)\quad \begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq 0 \end{aligned}$$

If $\bar{x}$ is feasible for (P), and
if $\bar{y}$ is feasible for (D),
then $c^T \bar{x} \leq b^T \bar{y}$.

Theorem (strong duality)

If x^* is optimal for (P), and if y^* is optimal for (D), then $c^T x^* = b^T y^*$.

Taking duals:

max	min
x_j free	constraint j $= c_j$
$x_j \geq 0$	constraint j $\geq c_j$
$x_j \leq 0$	constraint j $\leq c_j$
constraint i $\leq b_i$	$y_i \geq 0$
constraint i $\geq b_i$	$y_i \leq 0$
constraint i $= b_i$	y_i free

$\boxed{\text{not symmetric!}}$

Ex 1.

$$
(P)\quad
\begin{array}{llr}
\max & -3x_1 + x_2 + 2x_3 & \\
\text{s.t.} & 1x_1 + 2x_2 - 1x_3 \leq 8 & \leftarrow y_1 \\
& 4x_1 + 1x_2 - 1x_3 \geq 5 & \leftarrow y_2 \\
& 2x_1 + 2x_2 + 1x_3 = 1 & \leftarrow y_3 \\
& x_1 \geq 0, \quad x_2 \text{ free}, \quad x_3 \leq 0 &
\end{array}
$$

write dual:

$$
\begin{array}{ll}
\min & 8y_1 + 5y_2 + 1y_3 \\
\text{s.t.} & 1y_1 + 4y_2 + 2y_3 \geq -3 \\
& 2y_1 + 1y_2 + 2y_3 = 1 \\
& -1y_1 - 1y_2 + 1y_3 \leq 2 \\
& y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \text{ free}
\end{array}
$$

DEF. A constraint $\alpha^T x \ \substack{\geq \\ \leq \\ =}\ \beta$ is **tight at** $\bar{x}$ if $\alpha^T \bar{x} = \beta$.

Theorem (Complementary slackness)

Let $\bar{x}$ feasible for (P),
let $\bar{y}$ feasible for (D).
The following are equivalent:

1) $\bar{x}$ is <u>optimal</u> for (P), and
$\bar{y}$ is <u>optimal</u> for (D).

2) $c^T \bar{x} = b^T \bar{y}$

3) a) $\forall j$, $\bar{x}_j = 0$ **or** constraint j of (D) is tight at $\bar{y}$ (or both)

$\vdots$

e

b) $\forall i$, $\bar{y}_i = 0$ or constraint i of (P) is tight at $\bar{x}$ (or both)

Ex 1 (cont'd) verify that $\bar{x} = (0, 2, -3)$ is optimal for (P) and $\bar{y} = (0, -1, 1)$ is optimal for (D):

- $\boxed{\bar{x}_1 = 0}$ or $1\bar{y}_1 + 4\bar{y}_2 + 2\bar{y}_3 = -3$
- $\bar{x}_2 = 0$ or $\boxed{2\bar{y}_1 + 1\bar{y}_2 + 2\bar{y}_3 = 1}$
- $\bar{x}_3 = 0$ or $\boxed{-1\bar{y}_1 - 1\bar{y}_2 + 1\bar{y}_3 = 2}$
- $\boxed{\bar{y}_1 = 0}$ or $1\bar{x}_1 + 2\bar{x}_2 - 1\bar{x}_3 = 8$
- $\bar{y}_2 = 0$ or $\boxed{4\bar{x}_1 + 1\bar{x}_2 - 1\bar{x}_3 = 5}$
- $\bar{y}_3 = 0$ or $\boxed{2\bar{x}_1 + 2\bar{x}_2 + 1\bar{x}_3 = 1}$

$\Rightarrow$ $\bar{x}$ is optimal for (P)
$\bar{y}$ is optimal for (D).

Example: House Depot produces:

- hammers (fixed price 130)
  from 1.5 kg of steel, 1 rivet, 0.3 kg of plastic, and
- pliers (fixed price 100)
  from 1 kg of steel, 1 rivet, 0.5 kg of plastic.

Current stocks are:

27 kg of steel, 21 rivets, 9 kg of plastic.

Maximize income from current stocks.

(Assume that fractional hammers and pliers are ok)

VAR:

x_1 : hammers

x_2 : pliers

MODEL:

$$\max \; 130 \cdot x_1 + 100 \cdot x_2$$

$$\text{s.t.} \; 1.5\, x_1 + 1\, x_2 \leq 27 \quad \leftarrow \text{steel}$$

$$1\, x_1 + 1\, x_2 \leq 21 \quad \leftarrow \text{rivets}$$

$$0.3\, x_1 + 0.5\, x_2 \leq 9 \quad \leftarrow \text{plastic}$$

$$x_1 , \; x_2 \geq 0$$

EXAMPLE: House depot goes bankrupt. To buy their steel, rivets, and plastic, you need to propose a unit price for each. The liquidator will refuse your proposal if scrapping hammers or pliers for parts would be cheaper. Minimize your expense.

VAR:

y_1 : unit price for steel

y_2 : unit price for rivets

y_3 : unit price for plastic

MODEL:

$$\min \quad 27 \cdot y_1 + 21 \cdot y_2 + 9 \cdot y_3$$

(hammers) $\rightarrow$ $$\text{s.t.} \quad 1.5 \cdot y_1 + 1 \cdot y_2 + 0.3\, y_3 \geq 130$$

(pliers) $\rightarrow$ $$1 \cdot y_1 + 1 \cdot y_2 + 0.5\, y_3 \geq 100$$

$$y_1,\ y_2,\ y_3 \geq 0$$

In general, in a production problem subject to constrained resources, the optimal value of dual variables gives the shadow prices (or market/fair price) of the resources.

strong duality $\longrightarrow$

$$\text{income} = c^T x = b^T y = \sum_i b_i y_i = \text{market value of all resources}$$

Complementary slackness

$$\underbrace{\sum_i a_{ij}\, y_i > c_i}_{\substack{\text{market value of resources}\\ \text{for product } j\\ > \text{ product value}}} \quad \Rightarrow \quad \underbrace{x_j = 0}_{\text{do not produce!}}$$

$$\underbrace{\sum_j a_{ij}\, x_j < b_i}_{\text{Some resource unused}} \quad \Rightarrow \quad \underbrace{y_i = 0}_{\substack{\text{market price}\\ \text{is zero}}}$$

Additional resources:

$$z = \max \quad 130x_1 + 100x_2$$

$$1.5\,x_1 + 1\,x_2 \leq \cancel{27}\ 28$$

$$1\,x_1 + 1\,x_2 \leq 21$$

$$0.3\,x_1 + 0.5\,x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

optimal solution: $x = \cancel{(12, 9)}\ (14, 7)$

with value $z = \cancel{2460}\ 2520$

dual solution: $y = (\underline{60}, 40, 0)$

<u>note:</u> $2520 = 2460 + \underline{60} = 2460 + y_1$

THIS IS NOT ALWAYS THE CASE!

...BUT in general,

THEOREM:

Let $z = \max c^T x$ s.t. $Ax \leq b$, $x \geq 0$ (P) | (P') $z' = \max c^T x$, $Ax \leq b + \theta e_i$, $x \geq 0$

$$\begin{aligned} z = \max\ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned} \quad (P) \qquad\Bigg|\qquad (P') \quad \begin{aligned} z' = \max\ & c^T x \\ & Ax \leq b + \theta e_i \\ & x \geq 0 \end{aligned}$$

Then, if y^* is optimal for the dual of (P),

$$z' \leq z + \theta \cdot y_i^* .$$

__proof:__

Let

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \\ & y \geq 0 \end{aligned} \quad (D) \qquad\Bigg|\qquad (D') \quad \begin{aligned} \min\ & (b + \theta e_i)^T y \\ \text{s.t. } & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

We were given y^* optimal for (D).
Note that (D) and (D') have the same constraints, so y^* feasible for (D').

$\Rightarrow$ (P'), (D') both feasible

$\Rightarrow$ (P') has an optimal solution

$$z'^* \le b'^T y \quad \forall y \text{ feasible for (D')}$$

(by weak duality)

in particular

$$z'^* \le b'^T y^* = (b + \theta \cdot e_i)^T y^*$$
$$= b^T y^* + \theta \cdot y_i^*$$
$$= z^* + \theta \cdot y_i^*$$

$$\Rightarrow z'^* \le z^* + \theta \cdot y_i^*$$