Bases & duality

Consider the primal-dual pair:

$\min\ c^T x$ s.t. $Ax = b$ $x \geq 0$ (P)	$\max\ b^T y$ s.t. $A^T y \leq c$ (D) y free

Observe that (P) is in S.E.F., so we have a concept of a basis for (P). (D) is not in S.E.F.

Q: Can a basis $\mathcal{B}$ of (P) tell us something about (D)?

Intuition: Let $\mathcal{B}$ be a basis for (P).

$$\begin{array}{lll} \min & [c_{\mathcal{B}}^T \quad c_N^T]\, x & \\ \text{s.t.} & [\mathcal{B} \quad N]\, x = b & \text{(P)} \\ & x \geq 0 & \end{array} \qquad \Bigg| \qquad \begin{array}{lll} & \max & b^T y \\ \text{(D)} & \text{s.t.} & \begin{bmatrix} \mathcal{B}^T \\ N^T \end{bmatrix} y \leq \begin{bmatrix} c_{\mathcal{B}} \\ c_N \end{bmatrix} \\ & & y \text{ free} \end{array}$$

Let x^* be optimal for (P), with $x^* = \begin{bmatrix} \mathcal{B}^{-1} b \\ 0 \end{bmatrix}$,

let y^* be optimal for (D).

We know that $x^*_{\mathcal{B}} \geq 0$, assume $x^*_{\mathcal{B}} > 0$.

Then, by complementary slackness, the corresponding constraints in (D) are tight: $\mathcal{B}^T y^* = c_{\mathcal{B}}$.

Therefore, $y^* = (\mathcal{B}^T)^{-1} \cdot c_{\mathcal{B}}$

Definitions:

- $\bar{x}$ is the (primal) basic solution associated to $\mathcal{B}$ if $\bar{x}_{\mathcal{N}} = 0$ and $\bar{x}_{\mathcal{B}} = B^{-1} b$

(we already know this)

- $\bar{y}$ is the dual basic solution associated to $\mathcal{B}$ if $\bar{y} = (B^T)^{-1} \cdot c_{\mathcal{B}}$.
- $\mathcal{B}$ is (primal) feasible if $\bar{x}$ is feasible for (P)
- $\mathcal{B}$ is dual feasible if $\bar{y}$ is feasible for (D)

Theorem: Let $\mathcal{B}$ be a basis associated with primal solution $\bar{x}$ and dual solution $\bar{y}$.

The following are equivalent:

(1) $\bar{x}$ feasible for (P) <u>and</u> $\bar{y}$ feasible for (D)

(2) $\bar{x}$ is optimal for (P)

(3) $\bar{y}$ is optimal for (D)

proof:

(1) $\Leftrightarrow$ (2): Recall that $\bar{y} = (B^T)^{-1} \cdot c_B$

observe that: $\bar{y}$ feasible $\Leftrightarrow A^T \bar{y} \leq c$

$$\Leftrightarrow A^T (B^T)^{-1} c_B \leq c$$

$$\Leftrightarrow c - A^T (B^T)^{-1} c_B \geq 0$$

$$\Leftrightarrow c^T - c_B^T B^{-1} A \geq 0$$

$$\Leftrightarrow \bar{c} \geq 0$$

reduced costs

$$\begin{cases} \bar{x} \text{ feasible} \\ \bar{y} \text{ feasible} \end{cases} \Leftrightarrow \begin{cases} \bar{x} \text{ feasible} \\ \bar{c} \geq 0 \end{cases} \Leftrightarrow \bar{x} \text{ optimal}$$

(1) => (3):

Observe that: $c^T \bar{x} = c_B^T \bar{x}_B = c_B^T B^{-1} b$

$b^T \bar{y} = \bar{y}^T b = ((B^T)^{-1} c_B)^T b$

$= c_B^T B^{-1} b$

$=$

always true

$$\left\{\begin{array}{l} c^T \bar{x} = b^T \bar{y} \\ \bar{x} \text{ feasible} \\ \bar{y} \text{ feasible} \end{array}\right. \iff \left\{\begin{array}{l} \bar{x} \text{ optimal} \\ \bar{y} \text{ optimal}, \end{array}\right.$$

by strong duality.

(3) => (1): put (D) in SEF, then similar to (2) => (1).

Definition: Let $\mathcal{B}$ be a basis associated with primal basic solution $\bar{x}$ and dual basic solution $\bar{y}$.

$\mathcal{B}$ is optimal if $\bar{x}$ is optimal (or, equivalently, if $\bar{y}$ is optimal).

Sensitivity analysis

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

Suppose that $\mathcal{B}$ is an optimal basis for (P).

Sensitivity analysis answers the following Q:

> For what changes of A, b, c does $\mathcal{B}$ still give an optimal solution?

Changes to RHS:

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{st.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

$$\begin{aligned} \min \quad & c^T x \\ \text{st.} \quad & Ax = b + \theta \cdot e_i \\ & x \geq 0 \end{aligned} \qquad (P')$$

for $\theta \in \mathbb{R}$

Theorem: Let B be optimal for (P).
B is optimal for (P') if and only if

$$\underbrace{B^{-1} b}_{\bar{b}} + \theta \underbrace{B^{-1} e_i}_{i^{th} \text{ column of } B^{-1}} \geq 0$$

<u>proof.</u> B is optimal for (P)

$\Rightarrow$ $\quad \bar{b} = B^{-1}b \geq 0 \qquad (1)$

$\bar{c}^T = c^T - c_B^T B^{-1}A \geq 0 \qquad (2)$

B is optimal for (P') iff

$\bar{b}' = B^{-1}b' = B^{-1}(b + \theta e_i)$

$= B^{-1}b + \theta B^{-1}e_i \geq 0$

$\bar{c} = c'^T - c_B'^T B^{-1}A = c^T - c_B^T B^{-1}A$

$= \underbrace{\bar{c} \geq 0}_{\text{always holds, by (2)}}$

Observe that even if B stays optimal (P'),

$$\bar{x}'_B = B^{-1}b' = B^{-1}(b + \theta e_i) = \underbrace{B^{-1}b}_{\bar{x}_B} + \theta B^{-1}e_i \neq \bar{x}_B$$

However,

$$\bar{y}' = (B^T)^{-1}c'_B = (B^T)^{-1}c_B = \bar{y}$$

By strong duality $z' = c'^T\bar{x}' = b'^T\bar{y}'$

$$= (b+\theta e_i)^T \bar{y}$$

$$= b^T\bar{y} + \theta e_i^T \bar{y}$$

$$= z + \theta \bar{y}_i$$