changes to RHS:

Consider

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b + \theta . e_i \\ & x \geq 0 \end{aligned} \qquad (P')$$

for $\theta \in \mathbb{R}$

Theorem: Let $\mathcal{B}$ be optimal for (P).
$\mathcal{B}$ is optimal for (P') if and only if

$$\underbrace{B^{-1}b}_{\bar{b}} + \theta \underbrace{B^{-1}e_i}_{i^{th} \text{ column of } B^{-1}} \geq 0$$

[From lecture 12]

proof. $\mathcal{B}$ is optimal for (P)

$$\Rightarrow \quad \bar{b} = B^{-1}b \geq 0 \qquad (1)$$

$$\bar{c}^T = c^T - c_B^T B^{-1}A \geq 0 \qquad (2)$$

$\mathcal{B}$ is optimal for (P') iff

$$\bar{b}' = B^{-1}b' = B^{-1}(b + \theta e_i)$$

$$= B^{-1}b + \theta B^{-1} e_i \geq 0$$

$$\bar{c}' = c'^T - c_B'^T B^{-1}A = c^T - c_B^T B^{-1}A$$

$$= \underbrace{\bar{c} \geq 0}_{\text{always holds, by (2)}}$$

[From lecture 12]

Observe that even if B stays optimal (P'),

$$\bar{x}_B' = B^{-1}b' = B^{-1}(b+\theta e_i) = \underbrace{B^{-1}b}_{\bar{x}_B} + \theta B^{-1}e_i \neq \bar{x}_B$$

However,

$$\bar{y}' = (B^T)^{-1}c_B' = (B^T)^{-1}c_B = \bar{y}$$

By strong duality $z' = c'^T\bar{x}' = b'^T\bar{y}'$

$$= (b+\theta e_i)^T\bar{y}$$

$$= b^T\bar{y} + \theta e_i^T\bar{y}$$

$$= z + \theta\bar{y}_i$$

[from lecture 12]

EXAMPLE: House Depot produces:

- hammers (fixed price 130) from 1.5 kg of steel, 1 rivet, 0.3 kg of plastic, and
- pliers (fixed price 100) from 1 kg of steel, 1 rivet, 0.5 kg of plastic.

Current stocks are:

27 kg of steel, 21 rivets, 9 kg of plastic.

Maximize income from current stocks.

(Assume that fractional hammers and pliers are ok)

[From lecture 11]

VAR:

x_1 : hammers

x_2 : pliers

MODEL:

$$
\begin{array}{llcll}
\max & 130 \cdot x_1 & + & 100 \cdot x_2 & \\
\text{s.t.} & 1.5\, x_1 & + & 1\, x_2 \leq 27 & \leftarrow \text{steel} \\
 & 1\, x_1 & + & 1\, x_2 \leq 21 & \leftarrow \text{rivets} \\
 & 0.3\, x_1 & + & 0.5\, x_2 \leq 9 & \leftarrow \text{plastic} \\
 & x_1 & , & x_2 \geq 0 &
\end{array}
$$

[From lecture 11]

Example: hammers & pliers (cont'd)

S.E.F:

$$\min \begin{bmatrix} -130 & -100 & 0 & 0 & 0 \end{bmatrix} x$$

$$\text{s.t.} \begin{bmatrix} 1.5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0.3 & 0.5 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix}$$

$$x \geq 0$$

[simplex method $\Rightarrow$]

optimal basis: $B = \{1, 2, 5\}$

$$x^* = (12, 9, 0, 0, 0.9)$$

$$y^* = (-60, -40, 0)$$

$$z^* = -2460 \leftarrow -\text{profit}$$

optimal tableau:

$$\min \begin{bmatrix} 0 & 0 & 60 & 40 & 0 \end{bmatrix} x$$

$$\begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0.4 & -0.9 & 1 \end{bmatrix} x = \begin{bmatrix} 12 \\ 9 \\ 0.9 \end{bmatrix}$$

$$x \geq 0$$

optimal basis matrix:

$$B = \begin{bmatrix} 1.5 & 1 & 0 \\ 1 & 1 & 0 \\ 0.3 & 0.5 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 0 \\ 0.4 & -0.9 & 1 \end{bmatrix}$$

Q1: We change the available steel from 27 to $27+\theta$.
For what values of θ does the basis <u>remain</u> optimal?

$$b' = b + \theta \cdot e_1$$

$$= \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix} + \theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{b'} = B^{-1} \cdot b' = B^{-1} \cdot (b + \theta e_1)$$

$$= \underbrace{B^{-1} b}_{b} + \theta \underbrace{B^{-1} e_1}_{\text{first column of } B^{-1}}$$

$$= \begin{bmatrix} 12 \\ 9 \\ 0.9 \end{bmatrix} + \theta \begin{bmatrix} 2 \\ -2 \\ 0.4 \end{bmatrix} \geq 0$$

$$\begin{cases} 12 + \theta \cdot 2 \geq 0 \\ 9 + \theta \cdot (-2) \geq 0 \\ 0.9 + \theta \cdot 0.4 \geq 0 \end{cases}$$

implied by (3)

$$\Leftrightarrow \begin{cases} \theta \geq -6 & (1) \\ \theta \leq 4.5 & (2) \\ \theta \geq -2.25 & (3) \end{cases}$$

$$\Leftrightarrow \quad \theta \in [-2.25, \quad 4.5]$$

Definition: The set of all θ such that $\mathcal{B}$ stays optimal is called the **allowable range**.

Q2. What is the allowable range for plastic?

$$b' = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow e_3$$

$$\bar{b}' = \begin{bmatrix} 12 \\ 9 \\ 0.9 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \geq 0$$

← third column of B^{-1}

always true

$$\begin{cases} 12 + 0 \cdot \theta \geq 0 \\ 9 + 0 \cdot \theta \geq 0 \\ 0.9 + 1 \cdot \theta \geq 0 \end{cases} \iff \begin{cases} 12 \geq 0 \\ 9 \geq 0 \\ \theta \geq -0.9 \end{cases}$$

$$\theta \in [-0.9, +\infty[$$

Q3. A seller proposes 4 additional kg of steel for \$200. Should you accept? ______

Note: from Q1, $\theta = 4 \in [-2.25, 4.5]$,
i.e. 4 is in allowable range for steel.

$$z'^* = [-130 \;\; -100 \;\; 0 \;\; 0 \;\; 0] \cdot x^* = b'^T \cdot y'^*$$

$$= (b + \theta . e_1)^T y^*$$

$$= b^T y^* + \theta \, e_1^T y^*$$

$$= z^* + \theta \cdot y_1^*$$

$= -2460 - 60 \cdot 4$

$= -2460 - 240$

(-profit) increases by 240

profit increases by 240

$\longrightarrow$ ACCEPT!

Q4 Someone is willing to buy 2 kg of steel off our stock for \$130. Accept?

Note: $\theta = -2 \in [-2.25, 4.5]$

$\rightarrow$ basis stays the same

$$z'^{*} = z^{*} + \theta \cdot y_1^{*}$$
$$= -2460 + (-2) \cdot (-60)$$
$$= -2460 + 120$$

(-profit) increases by 120

profit decreases by $120 < 130$

$\rightarrow$ ACCEPT!

changes to objective function

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P)$$

$$\begin{aligned} \min \quad & (c + \theta . e_j)^T x \\ & A \; x = b \\ & x \geq 0 \end{aligned} \qquad (P')$$

for $\theta \in \mathbb{R}$

Theorem Let $\mathcal{B}$ be optimal for (P).

$\mathcal{B}$ is optimal for (P') if and only if

$$\begin{cases} \bar{c}_j + \theta \geq 0 & \text{if } j \notin \mathcal{B} \\[2ex] \bar{c}_N - \theta \underbrace{e_i^T B^{-1} N}_{\text{nonbasic part of } i^{th} \text{ row of optimal tableau.}} \geq 0 & \text{if } j \text{ is the } i^{th} \text{ basic column.} \end{cases}$$

proof $\quad B$ is optimal for (P) $\Rightarrow$

$$\bar{b} = B^{-1} b \geq 0 \qquad (1)$$

$$\bar{c}^T = c^T - c_B^T B^{-1} A \geq 0 \qquad (2)$$

B is optimal for (P') iff

$$\underbrace{\bar{b}' = B^{-1} b' = B^{-1} b = \bar{b} \geq 0}_{\text{always holds, by (1)}}$$

$$\bar{c}' = c'^T - c_B'^T B^{-1} A \geq 0$$

Case 1: $j \notin B$

$$\bar{c}' = (c + \theta e_j)^T - c_B^T B^{-1} A$$

$$= \underbrace{c^T - c_B B^{-1} A}_{\bar{c}} + \theta e_j^T$$

$$= \underbrace{\bar{c}^T}_{\geq 0} + \theta e_j^T \geq 0$$

$$\Leftrightarrow \quad \bar{c}_j + \theta \geq 0$$

Case 2: $j \in \mathcal{B}$, j is the i^{th} basic colu

$$\bar{c}' = (c + \theta e_j)^T - (c_{\mathcal{B}} + \theta e_i)^T B^{-1} A$$

$$= \underbrace{c^T - c_{\mathcal{B}}^T B^{-1} A}_{\bar{c}} + \theta e_j^T - \theta e_i^T B^{-1} A$$

$$= \bar{c}^T - \theta \Big(\underbrace{\underbrace{e_i^T B^{-1} A}_{i^{th}\text{ row of optimal tableau}} - e_j^T}_{i^{th}\text{ row of optimal tableau with a zero for basic variable}} \Big) \geq 0$$

$$\iff \quad \bar{c}_N^T - \theta \cdot e_i^T B^{-1} N \geq 0$$

Q5 What is the allowable range for c_1?

Note: $1 \in \mathcal{B}$, first basic variable

$$\bar{c}'^T = c'^T - c'_B B^{-1} A$$

$$= \underbrace{(c^T + \theta e_1^T)}_{\text{modifying } c_1} - \overbrace{(c_B^T + \theta e_1^T)}^{\text{first basic var.}} B^{-1} A$$

$$= c^T - c_B^T B^{-1} A - \theta(e_1^T B^{-1} A - e_1)$$

$$= \bar{c}^T - \theta([\cancel{1}\ 0\ 2\ -2\ 0] - [\cancel{1}\ 0\ 0\ 0\ 0])$$

$$= [0\ 0\ 60\ 40\ 0] - \theta[0\ 0\ 2\ -2\ 0] \geq 0$$

$$\Leftrightarrow \begin{cases} 0 - 0.\theta \geq 0 \\ 0 - 0.\theta \geq 0 \\ \boxed{\begin{matrix} 60 - 2.\theta \geq 0 \\ 40 + 2.\theta \geq 0 \end{matrix}} \\ 0 - 0.\theta \geq 0 \end{cases}$$

basic columns
trivially true

$$\Leftrightarrow \begin{cases} \theta \leq 30 \\ \theta \geq -20 \end{cases} \Leftrightarrow \theta \in [-20, 30]$$

Warning: in terms of original costs (max), we have

$$-\theta \in [-30, 20]$$