Previous lecture

Let $c' = c + \theta e_j$

$\mathcal{B}$ stays optimal if

$$\begin{cases} \bar{c}_j + \theta \geq 0 & \text{if } j \notin \mathcal{B} \\ \bar{c}_N - \theta e_i^T \mathcal{B}^{-1} N \geq 0 & \text{if } j \in \mathcal{B}, \ (*) \end{cases}$$

j is i^{th} basic var

(*) $\bar{c} - \theta(i^{th}$ tableau row with zero for $\bar{A}_{ij}) \geq 0$

Q6 What is the allowable range for c_5?

Note: x_5 is the 3^{rd} basic variable

3^{rd} tableau row, with zero for basic va

$$\begin{bmatrix} 0 & 0 & 60 & 40 & 0 \end{bmatrix} - \theta \begin{bmatrix} 0 & 0 & 0.4 & -0.9 & 0 \end{bmatrix} \geq 0$$

$$\begin{bmatrix} 0 & 0 & \underbrace{60-0.4\theta \quad 40+0.9\theta}_{\text{non basic variables}} & 0 \end{bmatrix} \geq 0$$

$$\begin{cases} 60 - 0.4\,\theta \geq 0 \\ 40 + 0.9\,\theta \geq 0 \end{cases} \Leftrightarrow \begin{cases} \theta \leq 150 \\ \theta \geq -44.44 \end{cases}$$

allowable range for c_5: $\theta \in [-44.44, 150]$

(for original) <u>max</u> problem: $\theta \in [-150, 44.44]$

Q7 Allowable range for c_3?

Note: x_3 nonbasic

$$\bar{c}_3 + \theta \geq 0$$

$$60 + \theta \geq 0$$

$$\theta \geq -60$$

allowable range: $\theta \in [-60, +\infty[$

for <u>max</u> problem, $\theta \in]-\infty, 60]$

Adding a new variable

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A x = b \\ & x \geq 0 \\ & x \in \mathbb{R}^n \end{aligned} \qquad (P)$$

and

$$\begin{aligned} \min \quad & [c^T \ \rho]\, x \\ \text{s.t.} \quad & [A \ \ g]\, x = b \\ & x \geq 0 \\ & x \in \mathbb{R}^{n+1} \end{aligned} \qquad (P')$$

Theorem Let $\mathcal{B}$ be optimal for (P).
$\mathcal{B}$ is optimal for (P') if and only if
$\bar{c}_{n+1} = p - y^{*T} f \geq 0$, where y^* is the
dual solution associated to $\mathcal{B}$.

proof $\mathcal{B}$ is optimal for (P) $\Rightarrow$

$$\bar{b} = \mathcal{B}^{-1} b \geq 0 \qquad (1)$$

$$\bar{c}^T = c^T - c_{\mathcal{B}}^T \mathcal{B}^{-1} A \geq 0 \qquad (2)$$

$\mathcal{B}$ is optimal for (P') iff

$$\overline{b'} = \mathcal{B}^{-1} b' = \mathcal{B}^{-1} b = \underbrace{\bar{b} \geq 0}_{\text{always holds, by (1)}}$$

$$\overline{c'}^T = c'^T - c_{\mathcal{B}}'^T \mathcal{B}^{-1} A' \geq 0$$

$$\iff \quad [c^T \;\; p] - c_B^T B^{-1} [A \;\; f] \geq 0$$

$$\iff \quad [c^T - c_B^T B^{-1} A \qquad p - c_B^T B^{-1} f] \geq 0$$

$$\iff \quad [\quad \bar{c} \qquad p - y^{*T} f] \geq 0$$

$\bar{c} \geq 0$ by (2) $\qquad$ since $y^* = B^{-1^T} c_B$

$$\iff \quad \begin{cases} \bar{c} \geq 0 & \text{(always true)} \\ p - y^{*T} f \geq 0 \end{cases}$$

Q8 Wrenches can be sold for a fixed price 100 and are produced from 2 kg steel, 1 rivet, 0.4 kg plastic. Should House Depot enter the wrench market?

$$\min \begin{bmatrix} -130 & -100 & 0 & 0 & 0 & -100 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 1.5 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0.3 & 0.5 & 0 & 0 & 1 & 0.4 \end{bmatrix} x = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix}$$

$$x \geq 0$$

Observation: Let $\mathcal{B}$ be optimal for (P).

(P) does not have x_6, so

$\mathcal{B}$ is optimal (P')

$\Longleftrightarrow$ x_6 is nonbasic at optimality

$\Longleftrightarrow$ its reduced cost $\bar{c}_6 \geq 0$

We have $\bar{c}_6 = p - y^{*T} f = (-100) - [-60 \ \ -90 \ \ 0] \cdot \begin{bmatrix} 2 \\ 1 \\ 0.9 \end{bmatrix}$

$= -100 - (-160) = 60 \geq 0$

$\Rightarrow$ $\mathcal{B}$ is optimal for (P')

x_6 is nonbasic at optimality

$x_6^* = 0$

Producing wrenches is <u>not</u> profitable

Q 9 What happens if $\mathcal{B}$ is <u>not</u> optimal for (P').

Changing nonbasic cost / add variable

- <u>one</u> reduced cost is negative
- apply simplex method, <u>starting at $\mathcal{B}$</u>
- modified variable enters the basis
- in practice, only one or few pivoti necessary

Changing basic cost

- typically, <u>few</u> reduced costs become negative
- Same as above

Changing RHS

- reduced costs remain ≥ 0
- typically, few components of $\bar{b}$ become negative
- could <u>not</u> start simplex method at $\mathcal{B}$.

$\Rightarrow$ <u>dual</u> simplex method

Dual simplex method

(primal) simplex method

- maintains $\bar{b} \geq 0$
- achieves $\bar{c} \geq 0$

dual simplex method

- maintains $\bar{c} \geq 0$
- achieves $\bar{b} \geq 0$

Example tableau:

$$
\begin{array}{llllllll}
\min & & & & 3x_4 & + & 2x_5 & \\
\text{s.t.} & x_1 & & & -2x_4 & - & x_5 & = -2 \\
& & x_2 & & -x_4 & + & x_5 & = 1 \\
& & & x_3 & & + & 2x_5 & = 1
\end{array}
$$

- A negative variable will leave the basis

  $\to$ x_1 leaves $\mathcal{B}$, x_e enters for $e \in \{4,5\}$

- In the next tableau, we need $\bar{c}'_j \geq 0 \quad \forall j$

  $\bar{c}'_e = 0$

- Can be achieved by adding λ times the first row to the objective function (i.e. adding constant $\lambda \cdot (-2)$).
- we will have

$$(\text{constant} +) \min \ \lambda x_1 \qquad + (3-2\lambda) x_4 + (2-\lambda) x_5$$

- $\bar{c}_1' = \bar{c}_1 + \lambda = 0 + \lambda = \lambda \geq 0$ (*)
- $\bar{c}_e' = \bar{c}_e + \lambda \bar{A}_{1e} = 0$

$$\Rightarrow \lambda = \frac{\bar{c}_e}{-\bar{A}_{1e}}$$

$$\Rightarrow \text{by } (*), \quad \lambda = \frac{\bar{c}_e}{-\bar{A}_{1e}} \geq 0 \Rightarrow \bar{A}_{1e} < 0$$

≥ 0

- $\forall j \neq e, \quad \bar{c}'_j = \bar{c}_j + \lambda \bar{A}_{1j} \geq 0$

$$\Rightarrow \quad \underbrace{\lambda}_{\substack{\geq 0 \\ \text{by } (*)}} \bar{A}_{1j} \geq \underbrace{- \underbrace{\bar{c}_j}_{\geq 0}}_{\leq 0}$$

- always true if $\bar{A}_{1j} \geq 0$
- if $\bar{A}_{1j} < 0, \quad \lambda \leq - \dfrac{\bar{c}_j}{\bar{A}_{1j}}$
- we need $\lambda = \min \underbrace{\left\{ \dfrac{\bar{c}_j}{-\bar{A}_{1j}} \;\middle|\; \bar{A}_{1j} < 0 \right\}}_{\text{ratio test for dual simplex method}}$