In the example:

$$\lambda = \min \left\{ \underbrace{\frac{3}{-(-2)}}, \frac{2}{-(-1)} \right\}$$

$$\Rightarrow \lambda = \frac{3}{2}, \quad x_4 \text{ enters the basis}$$

$$\bar{c}' = \bar{c} + \lambda \underset{\text{first tableau row}}{\bar{A}_1} = \left[\underbrace{\tfrac{3}{2}}_{\substack{\bar{c}'_1 \geq 0 \\ \text{ok}}} \quad 0 \quad 0 \quad \underbrace{0}_{\substack{\bar{c}'_4 = 0 \\ \text{ok}}} \quad \underbrace{\tfrac{1}{2}}_{\substack{\geq 0 \\ \text{ok}}} \right]$$

Dual simplex method:

Given a basis $\mathcal{B}$ such that $\bar{c} \geq 0$.

- choose pivot row i : $\bar{b}_i < 0$
- leaving variable is basic variable in row i
- choose entering variable : ratio test:

$$e = \operatorname*{argmin}_j \left\{ \frac{\bar{c}_j}{-\bar{A}_{ij}} \;\middle|\; \bar{A}_{ij} < 0 \right\}$$

- pivot

The algorithm ends when

- there is no leaving variable (optimality)
- there is no entering variable
  (dual unboundedness → primal infeasibility)

Example:

$$\min \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \end{bmatrix} x$$

$$\text{s.t.} \begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ -1 & -1 & 2 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x \geq 0$$

<u>First iteration:</u>

$$\mathcal{B} = \{4, 5\}, \qquad \bar{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \leftarrow \text{pivot row } i=2$$

leaving var: x_5

ratio test: $$e = \underset{j}{\operatorname{argmin}} \left\{ \frac{\bar{c}_j}{-\bar{A}_{ij}} \;\middle|\; \bar{A}_{ij} < 0 \right\}$$

$$= \underset{j}{\operatorname{argmin}} \left\{ \frac{1}{-(-1)}, \frac{2}{-(-1)}, \cancel{\frac{1}{-(2)}} \right\}$$

$$= \arg\min_{j} \left\{ \underset{j=1}{1}, \ \underset{j=2}{2} \right\}$$

$$= 1$$

$\hookrightarrow$ entering variable : x_1

<u>Second iteration:</u>

$$\mathcal{B} = \{4, 1\}$$

$$\min \begin{bmatrix} 0 & 1 & 3 & 0 & 1 \end{bmatrix} x$$

$$\text{s.t.} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x \geq 0$$

$$x^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

optimal

PART $\underline{\text{IV}}$: Solving IPs

Definition:

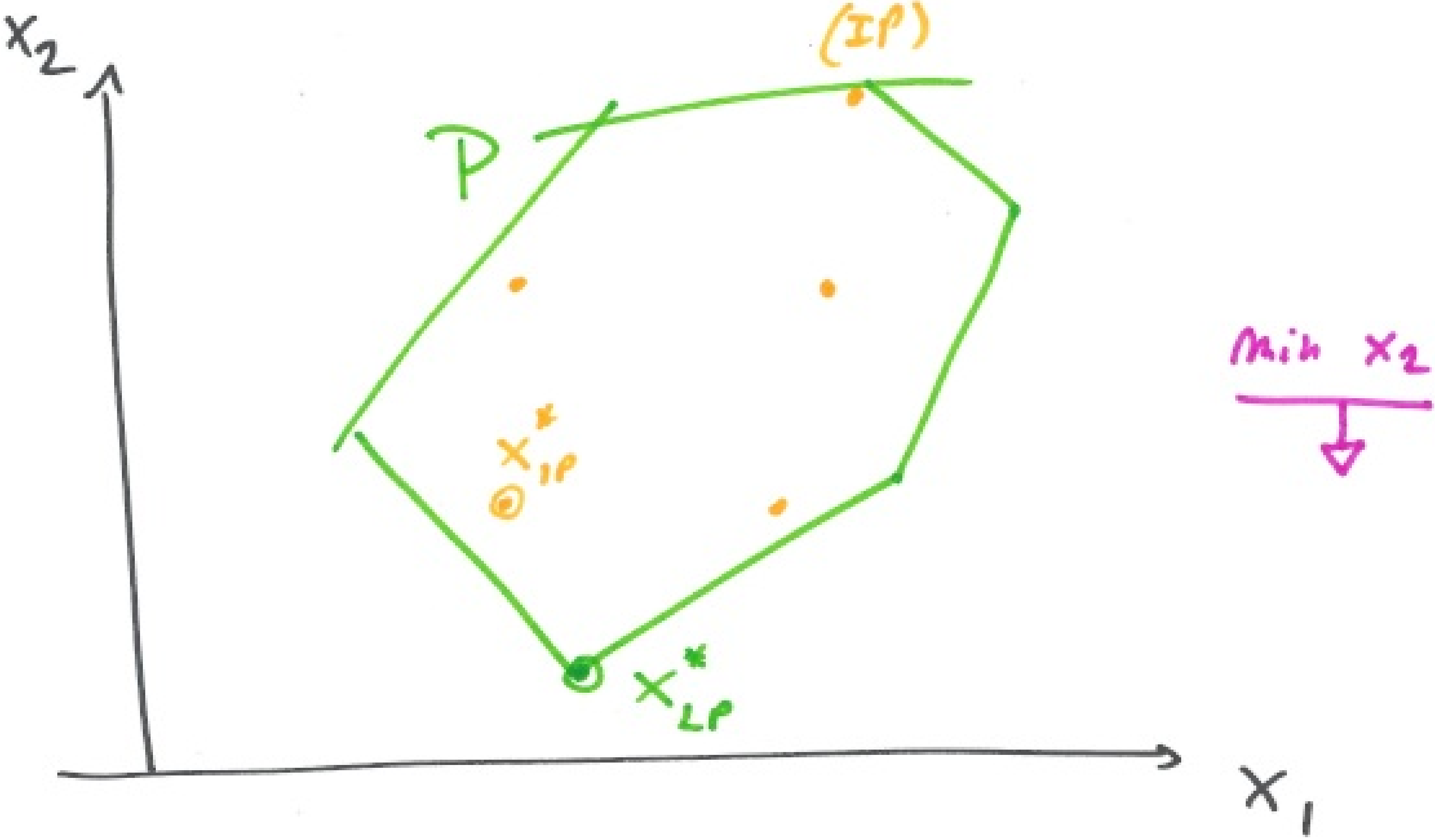
Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a polyhedron.

- An **integer programming** problem is a problem of the form:

$$z^*_{IP} = \begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in \mathcal{P} \\ & x_j \in \mathbb{Z}, \ \forall j \in J \subseteq \{1, \dots, n\} \end{array} \qquad \text{(IP)}$$

- The **LP relaxation** of (IP) is a linear programming problem:

$$z^*_{LP} = \min \; c^T x \quad \text{s.t.} \; x \in P \qquad \text{(LP)}$$



<u>Note</u>: we always assume that A, b, c are <u>rational</u>.

Theorem 1: $z^*_{LP} \leq z^*_{IP}$

proof: By definition of optimality

$$z^*_{LP} \leq c^T x \ , \qquad \forall x \in P$$

Observe that the feasible solutions of (IP) are a subset of P.

$\Rightarrow \quad z^*_{LP} \leq c^T x \ , \ \forall x$ feasible for (IP)

In particular, $z^*_{LP} \leq c^T x^*_{IP} = z^*_{IP}$.

Theorem 2: If x^*_{LP} is optimal for (LP) and $x^*_{LP_j} \in \mathbb{Z}$, $\forall j \in J$, then x^*_{LP} is optimal for (IP).

proof Again $c^T x^*_{LP} \leq c^T x$ $\forall x$ feasible for (IP).

Here, x^*_{LP} is feasible for (IP).

$\Rightarrow$ optimal for (IP).

Branching method:

Given (IP), consider its LP relaxation (LP).

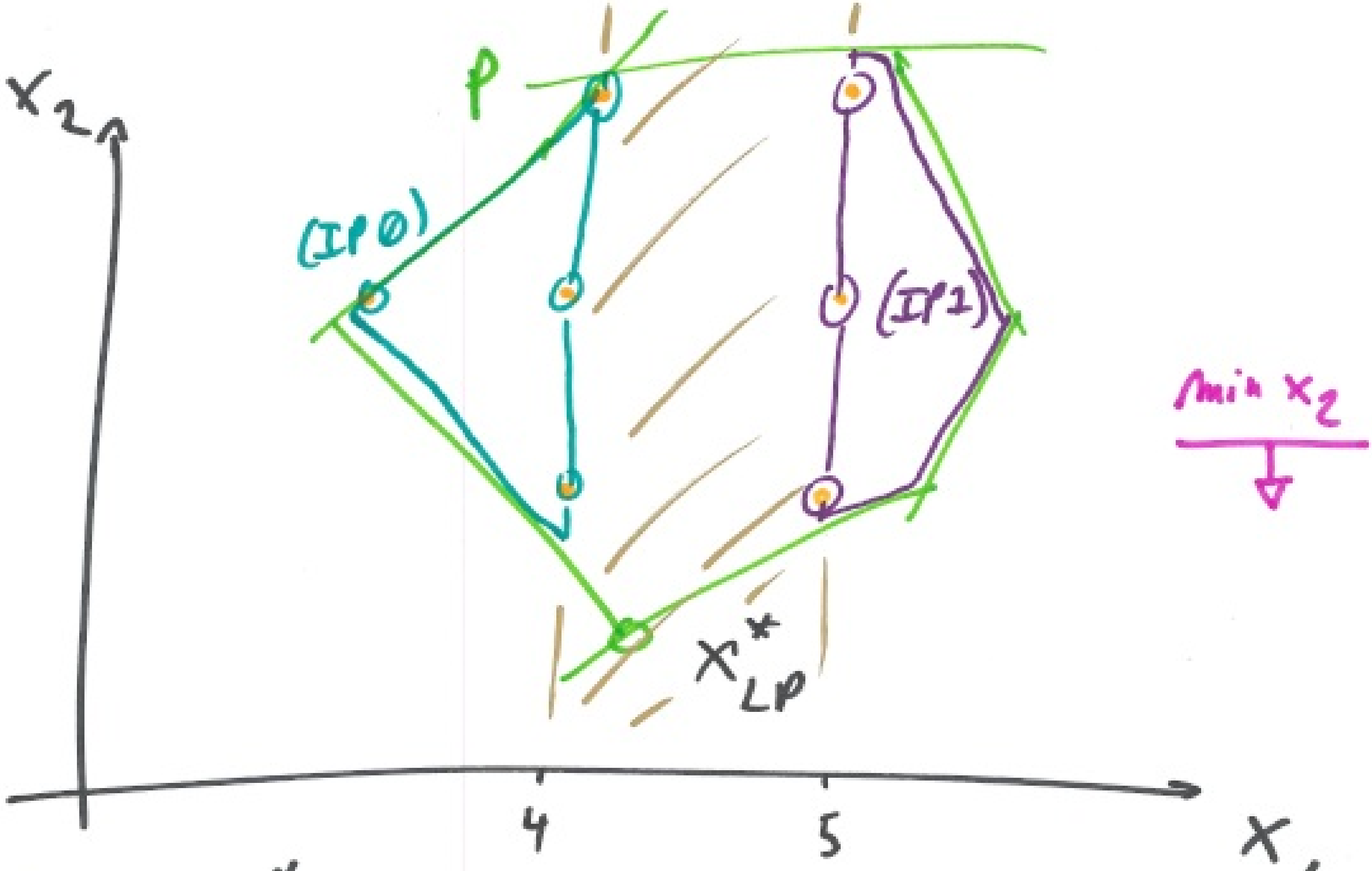
- if (LP) is unbounded, (IP) is infeasible or unbounded.
- if (LP) is infeasible, (IP) is infeasible
- otherwise, let x^*_{LP} be optimal for (LP).
  - if $x^*_{LP_j} \in \mathbb{Z}$, $\forall j \in J$, then return x^*_{LP} (optimal for (IP))
  - otherwise let $k \in J$: $x^*_{LP_k} \notin \mathbb{Z}$

and solve

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x \in P \\ & x_j \in \mathbb{Z}, \quad \forall j \in J \qquad (IP0) \\ & x_k \leq \lfloor x^*_{LPk} \rfloor \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ & x \in P \\ & x_j \in \mathbb{Z}, \quad \forall j \in J \qquad (IP1) \\ & x_k \geq \lceil x^*_{LPk} \rceil \end{aligned}$$

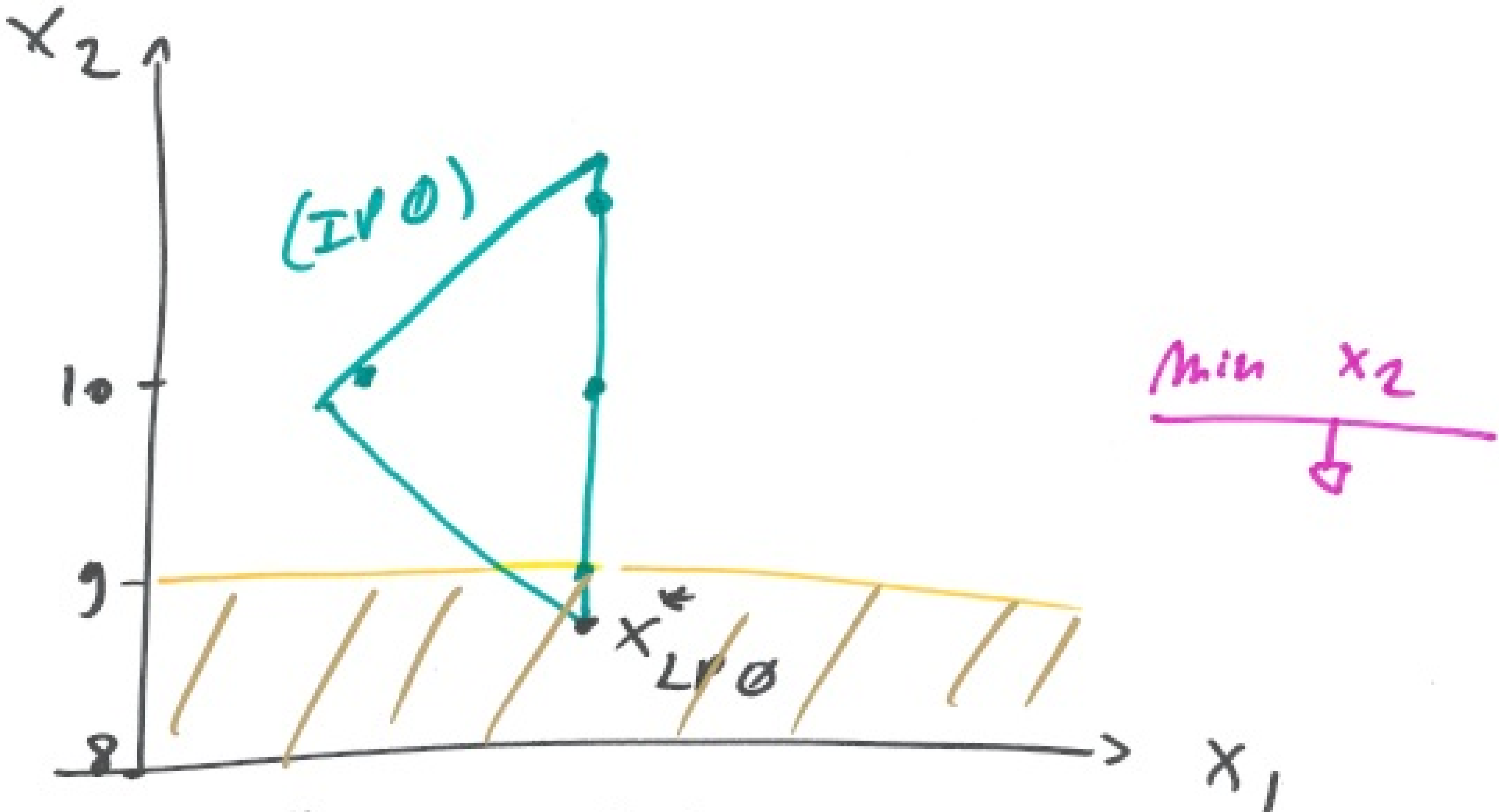
return the best of the solutions to (IP0) and (IP1).



$x^*_{LP,1} = 4.2$

(IP0) : $x_1 \leq 4$

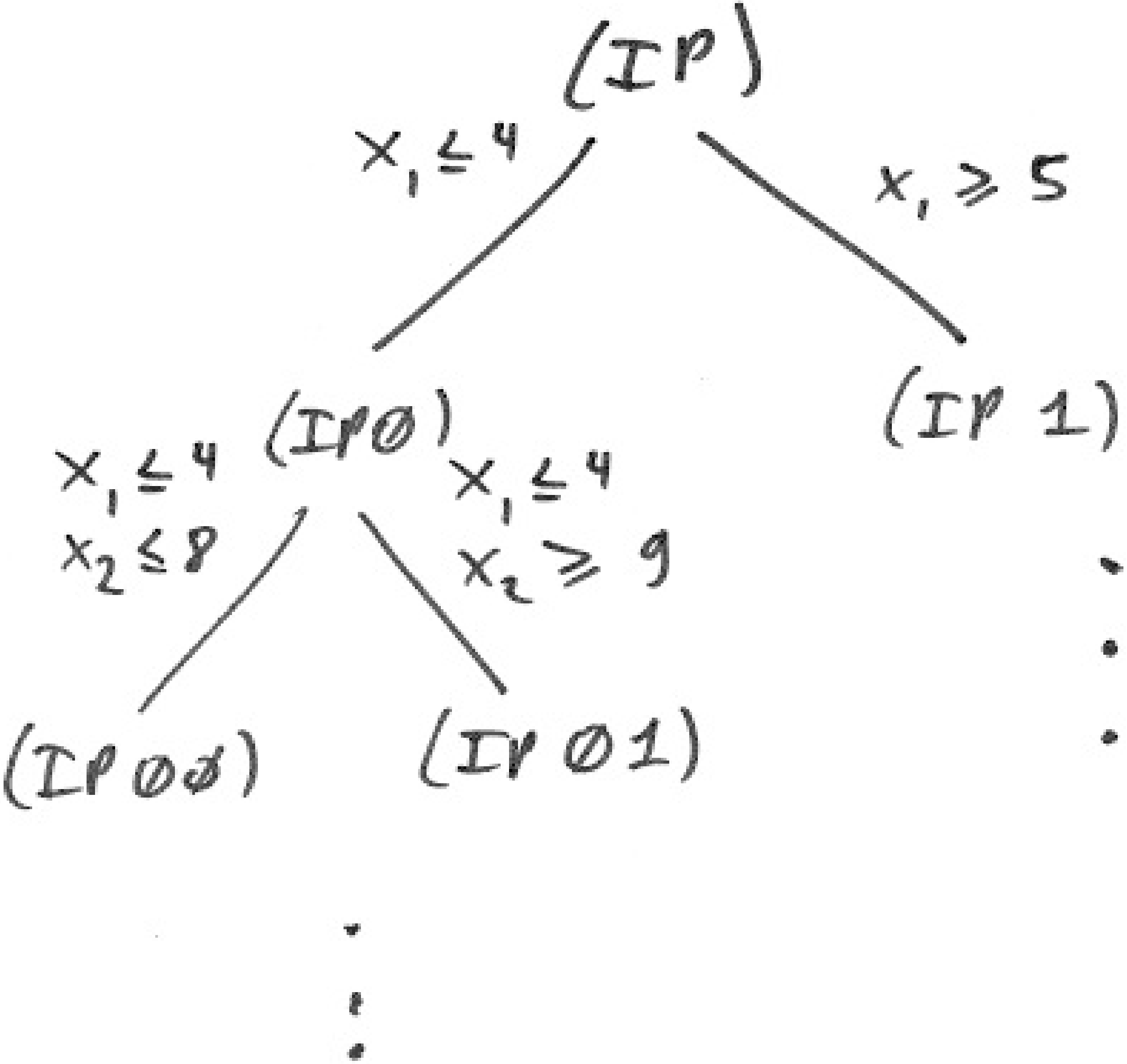
(IP1) : $x_1 \geq 5$



$x^*_{LP0,2} = 8.8$

(IP00) : $x_1 \le 4$, $x_2 \le 8$

(IP01) : $x_1 \le 4$, $x_2 \ge 9$


(IP)
x1 ≤ 4
x1 ≥ 5
(IP0)
(IP1)
x1 ≤ 4
x2 ≤ 8
x1 ≤ 4
x2 ≥ 9
(IP00)
(IP01)

Branch and bound method

initialize $\bar{z} = +\infty$
Solve (IP)
optimal solution $\bar{x}$ with value $\bar{z}$.

Solve $(\widetilde{IP})$:

if $(L\tilde{P})$ is infeasible or unbounded,
return.

let $\tilde{x}$ be optimal for $(L\tilde{P})$ and
$\tilde{z} = c^T \tilde{x}$.

if $\tilde{x}$ is feasible for $(\tilde{IP})$

  if $\tilde{z} < \bar{z}$

    $\bar{z} := \tilde{z}$

    $\bar{x} := \tilde{x}$

  return

if $\tilde{z} \geqslant \bar{z}$

  (we know that all solutions to $(\tilde{IP})$ are no better than $\bar{x}$)

  return

let $k \in J$: $\tilde{x}_k \notin \mathbb{Z}$

Solve $(\tilde{IP} \wedge x_k \leq \lfloor \tilde{x}_k \rfloor)$

Solve $(\tilde{IP} \wedge x_k \geq \lceil \tilde{x}_k \rceil)$

Remark: We need to solve one LP at each iteration.

However, if we add constraints

$x \geq l$ and $x \leq u$:

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & A x \geq b \\
& x \geq l \\
& x \leq u \\
& x_j \in \mathbb{Z}, \quad \forall j \in J
\end{aligned}
$$

then all LPs are identical, except for the RHS l and u.

$\Rightarrow$ Dual simplex method