Knapsack problem

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a^T x \le b \\ & 0 \le x \le u \\ & x \in \mathbb{Z}^n \end{aligned} \qquad (K)$$

where $b > 0$, $a_j, c_j, u_j > 0, \ \forall j$

How to solve the LP relaxation of (K)?

Example 1:

$$\max \quad 50\, x_1 + 10\, x_2 + 2 x_3 + \frac{1}{50} x_4$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 \leq 14$$

$$0 \leq x_1 \leq 4, \quad 0 \leq x_2 \leq 5, \quad 0 \leq x_3 \leq 10, \quad 0 \leq x_4 \leq 50$$

$$\cancel{x \in \mathbb{Z}^4}$$

Solution: x_1 has best value $\rightarrow$ take $x_1^* = 4$

x_2 is next best $\rightarrow$ take $x_2^* = 5$

then $x_3 \rightarrow$ set $x_3^* = 14 - 4 - 5 = 5$

set $x_4^* = 0$

Example 2:

$$\max\ 100x_1 + 50x_2 + 20x_3 + 1x_4$$

$$\text{s.t.}\quad 2x_1 + 5x_2 + 10x_3 + 50x_4 \le 14$$

$$0 \le x_1 \le 2,\quad 0 \le x_2, x_3, x_4 \le 1$$

$$\cancel{x \in \mathbb{Z}^n}$$

Solution: Let $y_j := a_j \cdot x_j$.

In other words, $x_1 = \frac{1}{2} y_1$, $x_2 = \frac{1}{5} y_2$,

$x_3 = \frac{1}{10} y_3$, $x_4 = \frac{1}{50} y_4$

We obtain:

$$\max \quad 50 y_1 + 10 y_2 + 2 y_3 + \frac{1}{50} y_4$$

$$\text{s.t.} \quad y_1 + y_2 + y_3 + y_4 \leq 14$$

$$0 \leq y_1 \leq 4, \quad 0 \leq y_2 \leq 5, \quad 0 \leq y_3 \leq 10, \quad 0 \leq y_4 \leq 50$$

$\longrightarrow$ this is exactly example 1 observe that $\frac{100}{2} \geq \frac{50}{5} \geq \frac{20}{10} \geq \frac{1}{50}$

$\rightarrow$ solve greedily like example 1:

Take $x_1 \leq 2$: $\frac{14}{2} = 7 \not\leq 2 \rightarrow x_1^* = 2$

$x_2 \leq 1$: $\frac{14 - 2.2}{5} = 2 \not\leq 1 \rightarrow x_2^* = 1$

$x_3 \leq 1$: $\frac{14 - 2.2 - 5.1}{10} = \frac{1}{2} \leq 1 \rightarrow x_3^* = \frac{1}{2}$

$x_4 \leq 1$: $\frac{14 - 2.2 - 5.1 - 10.\frac{1}{2}}{50} = 0 \leq 1 \rightarrow x_4^* = 0$

Greedy algorithm for continuous knapsack

Sort variables such that $\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \dots \geq \frac{c_n}{a_n}$

For $j = 1, \dots, n$

Compute $\lambda := \dfrac{b - \sum_{k=1}^{j-1} a_k x_k^*}{a_j}$

If $\lambda > u_j$, set $x_j^* := u_j$

If $0 \leq \lambda \leq u_j$, set $x_j^* := \lambda$

(in particular, if $\lambda = 0$, $x_j^* := 0$)

Reminder: branch-and-bound method.

Create a branch-and-bound tree.

At every node of the tree:

- Solve LP relaxation of subproblem
  - infeasible → done
  - compare to best integer solution found
    - not better → done
  - if integer, set as new best solution
  - if fractional, branch on a fractional component, create 2 "child" nodes.

Example 3:

$$\max \quad 2x_1 + 2x_2 + 6x_3 + 5x_4$$
$$\text{s.t.} \quad 7x_1 + 2x_2 + 8x_3 + 6x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

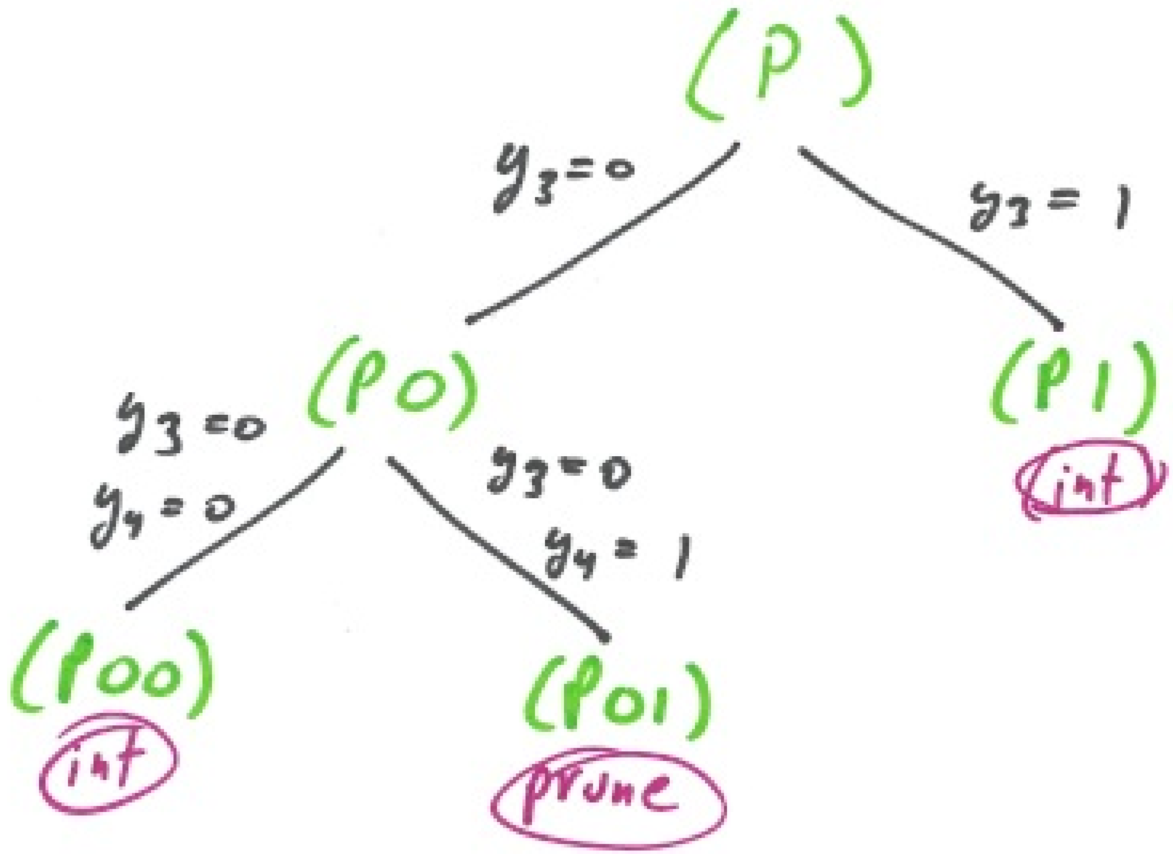
Solution:

$$\frac{2}{2} \geq \frac{5}{6} \geq \frac{6}{8} \geq \frac{2}{7}$$

Let $y_1 := x_2, \quad y_2 := x_4, \quad y_3 := x_3, \; y_4 := x_1$

$$\max \quad 2y_1 + 5y_2 + 6y_3 + 2y_4$$
(P) $$\text{s.t.} \quad 2y_1 + 6y_2 + 8y_3 + 7y_4 \leq 10$$
$$y_1, y_2, y_3, y_4 \in \{0,1\}$$

(node)	fixed	LP relaxation	best intege
(P)		$\tilde{y} = [1 \;\; 1 \;\; \frac{1}{4} \;\; 0]$, $\tilde{z} = 8.5$	
(P0)	$y_3 = 0$	$\tilde{y} = [1 \;\; 1 \;\; 0 \;\; \frac{2}{7}]$, $\tilde{z} = 7 + \frac{4}{7}$	
(P00)	$y_3 = 0$, $y_4 = 0$	$\tilde{y} = [\underline{1 \;\; 1 \;\; 0 \;\; 0}]$, $\tilde{z} = 7$ int	$\bar{z} = 7$
(P01)	$y_3 = 0$, $y_4 = 1$	$\tilde{y} = [1 \;\; \frac{1}{6} \;\; 0 \;\; 1]$, $\tilde{z} = 4 + \frac{5}{6} \leq 7$	
(P1)	$y_3 = 1$	$\tilde{y} = [\underline{1 \;\; 0 \;\; 1 \;\; 0}]$, $\tilde{z} = 8$ int	$\bar{z} = 8$


(P)
$y_3=0$
$y_3=1$
(P0)
(P1)
int
$y_3=0$
$y_4=0$
$y_3=0$
$y_4=1$
(P00)
int
(P01)
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Example 4: $\max\ 6x_1 + 9x_2 + 5x_3 + 3x_4 + 7x_5$

$$7x_1 + 25x_2 + 10x_3 + 25x_4 + 9x_5 \leq 42$$

$$x_1, \ldots, x_5 \in \{0,1\}$$

Solution: first, we sort the variables:

$$x_1 \;,\; x_5 \;,\; x_3 \;,\; x_2 \;,\; x_4$$

$$\frac{6}{7} \geq \frac{7}{9} \geq \frac{5}{10} \geq \frac{9}{25} \geq \frac{3}{25}$$

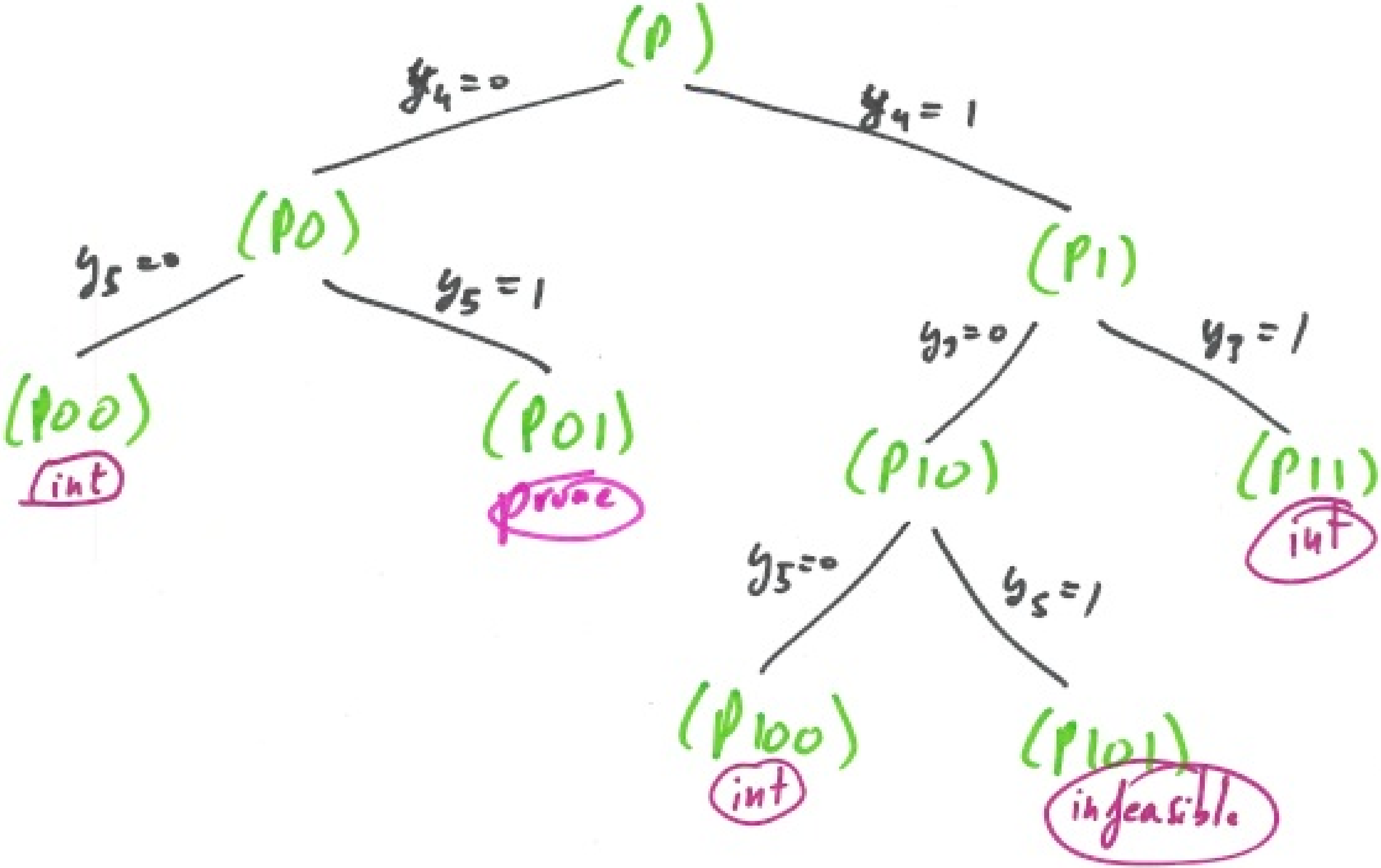
the problem becomes:

$$(P)\quad \begin{aligned} \max\ & 6y_1 + 7y_2 + 5y_3 + 9y_4 + 3y_5 \\ \text{s.t.}\ & 7y_1 + 9y_2 + 10y_3 + 25y_4 + 25y_5 \leq 42 \\ & y_j \in \{0,1\} \quad \forall j \end{aligned}$$

node	fixed	LP relaxation	best
(P)		$\tilde{y} = [1\ 1\ 1\ 0.64\ 0]$, $\tilde{z} = 23.76$	
(P0)	$y_4 = 0$	$\tilde{y} = [1\ 1\ 1\ 0\ 0.69]$, $\tilde{z} = 19.92$	
(P00)	$y_4 = 0$, $y_5 = 0$	$\tilde{y} = [1\ 1\ 1\ 0\ 0]$, $\tilde{z} = 18$ int	$\bar{z} = 18$
(P01)	$y_4 = 0$, $y_5 = 1$	$\tilde{y} = [1\ 1\ 0.1\ 0\ 1]$, $\tilde{z} = 16.5 \leq 18$	
(P1)	$y_4 = 1$	$\tilde{y} = [1\ 1\ 0.1\ 1\ 0]$, $\tilde{z} = 22.5$	
(P10)	$y_4 = 1$, $y_3 = 0$	$\tilde{y} = [1\ 1\ 0\ 1\ 0.04]$, $\tilde{z} = 22.12$	
(P100)	$y_4 = 1$, $y_3 = 0$, $y_5 = 0$	$\tilde{y} = [1\ 1\ 0\ 1\ 0]$, $\tilde{z} = 22$ int	$\bar{z} = 22$
(P101)	$y_4 = 1$, $y_3 = 0$, $y_5 = 1$	$\tilde{y} = [?\ ?\ 0\ 1\ 1]$ infeasible	
(P11)	$y_4 = 1$, $y_3 = 1$	$\tilde{y} = [1\ 0\ 1\ 1\ 0]$, $\tilde{z} = 20$ int	

$$y^* = (1, 1, 0, 1, 0), \quad z^* = 22$$

$$x^* = (1, 0, 0, 1, 1) \quad z^* = 22$$


(P)
y4=0
y4=1
(P0)
(P1)
y5=0
y5=1
y7=0
y7=1
(P00)
int
(P01)
prune
(P10)
(P11)
int
y5=0
y5=1
(P100)
int
(P101)
infeasible