Totally unimodular matrices

Definition Given a square matrix $A \in \mathbb{R}^{n \times n}$, its cofactor matrix $C \in \mathbb{R}^{n \times n}$ satisfies $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the matrix obtained by removing row i and column j from A.

Example: If $A = \begin{bmatrix} 7 & 5 & 11 \\ 4 & 0 & 9 \\ 1 & 3 & -1 \end{bmatrix}$,

then

$$C_{12} = (-1)^{1+2} \cdot \det \begin{bmatrix} 4 & 9 \\ 1 & -1 \end{bmatrix}$$

$$= (-1) \cdot (4 \cdot (-1) - 1 \cdot 9) = 13$$

Theorem 1 [Laplace expansion]

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij} \quad \text{for any row } i$$

$$= \sum_{i=1}^{n} A_{ij} C_{ij} \quad \text{for any column } j$$

Example: If $A = \begin{bmatrix} 7 & 5 & 11 \\ 4 & 0 & 9 \\ 1 & 3 & -1 \end{bmatrix}$,

then $\det(A) = 7 \cdot C_{11} + 5 \cdot C_{12} + 11 \cdot C_{13}$

$$= 7 \cdot (-1)^{1+1} \det \begin{bmatrix} 0 & 9 \\ 3 & -1 \end{bmatrix} + 5 \cdot 13 + 11 \cdot (-1)^{1+3} \det \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= 7(-27) + 5 \cdot 13 + 11 \cdot 12$$

$$= -189 + 65 + 132 = 8$$

Theorem 2 [Cramer's rule]

$$A^{-1} = \frac{1}{\det(A)} \cdot C^T$$

Definition A square, integer matrix $B \in \mathbb{Z}^{m \times m}$ is **unimodular** if $\det(B) = -1$ or $\det(B) = 0$ or $\det(B) = +1$.

Theorem 3 Let $B \in \mathbb{Z}^{m \times m}$ be unimodular and invertible (i.e. $\det(B) = -1$ or $\det(B) = +1$).

Then (1) $B^{-1} \in \mathbb{Z}^{m \times m}$,

(2) B^{-1} is unimodular

(3) $B^{-1} d \in \mathbb{Z}^m$, $\forall d \in \mathbb{Z}^m$.

proof: (1) Observe that by Theorem 1 (Laplace expansion), $\forall A \in \mathbb{Z}^{n \times n}$, $\det(A) \in \mathbb{Z}$. Thus the cofactor matrix of B satisfies $C \in \mathbb{Z}^{m \times m}$. By Theorem 2 (Cramer's rule), $B^{-1} = \frac{1}{\det(B)} \cdot C^T \in \mathbb{Z}^{m \times m}$ since $\det(B)$ is -1 or $+1$.

(2) $\det(B^{-1}) = [\det(B)]^{-1} = -1$ or $+1$

since B is unimodular.

(3) Since B^{-1} is integer, $B^{-1}d$ is

integer as well.

Theorem 4 Consider the polyhedron in SEF
$P = \{ x \in \mathbb{R}^n : Ax = b \text{ and } x \geq 0 \}$
where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.
If every $m \times m$ submatrix of A is unimodular, then all vertices of P are integral.

proof: Vertices of P are its basic feasible solutions, which are computed by setting $x_N = 0$ and $x_B = B^{-1}b$, where $\mathcal{B}$ is a basis of P. Since $\mathcal{B}$ is a basis, B is an $m \times m$ invertible submatrix of A, thus by hypothesis B is invertible and unimodular. By Theorem 3, $x_B = B^{-1}b \in \mathbb{Z}^m$.

Definition An integer matrix $A \in \mathbb{Z}^{m \times n}$ is **totally unimodular** (TU) if every square submatrix of A is unimodular (all of them, not just the $m \times m$ ones).

note: in particular 1×1 submatrices of A are unimodular, so $A_{ij} \in \{-1, 0, +1\}$.

Theorem 5 Let $A \in \mathbb{Z}^{m \times n}$ be a TU matrix.
The following operations yield another TU matrix

(a) multiplying a row or a column by -1.

(b) permuting the rows or columns

(c) transposing the matrix

(d) adding a row or column that is zero everywhere except at most one entry $+1$ or -1.

(e) adding a row or column of A.

<u>proof</u> Consider all submatrices M of A. After the operation ((a),(b),..., (e)), M is either unchanged, or the same operation was applied to M. In that case, let us observe how it affects $\det(M)$.

(a) $\det(M)$ is multiplied by -1

(b) only the sign of $\det(M)$ is affected

(c) $\det(M^T) = \det(M)$

(d) $\det([M \mid 0]) = 0$

$\det([M \mid e_i]) = \det\left(\left[\begin{array}{c|c} M & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]\right)$

$= -1 \cdot$ or $+1 \cdot \det(\text{submatrix of } M) = -1$ or $+1$

$\det([M \mid -e_i]) = -1$ or $+1$ (*)

(*) by the Laplace expansion.

(e) $\det([M' \mid M'e_i]) = 0$, where $M' \in \mathbb{Z}^{k \times (k-1)}$

Corollary Let $A \in \mathbb{Z}^{m \times n}$ be a TU matrix,

(1) any submatrix of A is TU

(2) A^T is TU

(3) $[A \mid I]$ and $\left[\frac{A}{I}\right]$ are TU

(4) $[A \mid A]$ and $\left[\frac{A}{A}\right]$ are TU

Theorem 6: Consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. If A is TU, then all vertices of P are integer.

proof: Let us add slack variables:

$$P' = \left\{ \begin{bmatrix} x \\ s \end{bmatrix} \in \mathbb{R}^{n+m} : [A \mid I] \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b \right\}.$$

We know that $[A \mid I]$ is TU, so all its $m \times m$ submatrices are unimodular. By Theorem 4, all vertices $\begin{bmatrix} \bar{x} \\ \bar{s} \end{bmatrix}$ of P' are integral. In particular, $\bar{x}$ is integral, so all vertices of P are integral.

Note: In particular, if we optimize over P with the <u>simplex method</u>, we always obtain a vertex of P, hence an integer solution.