Previous lecture

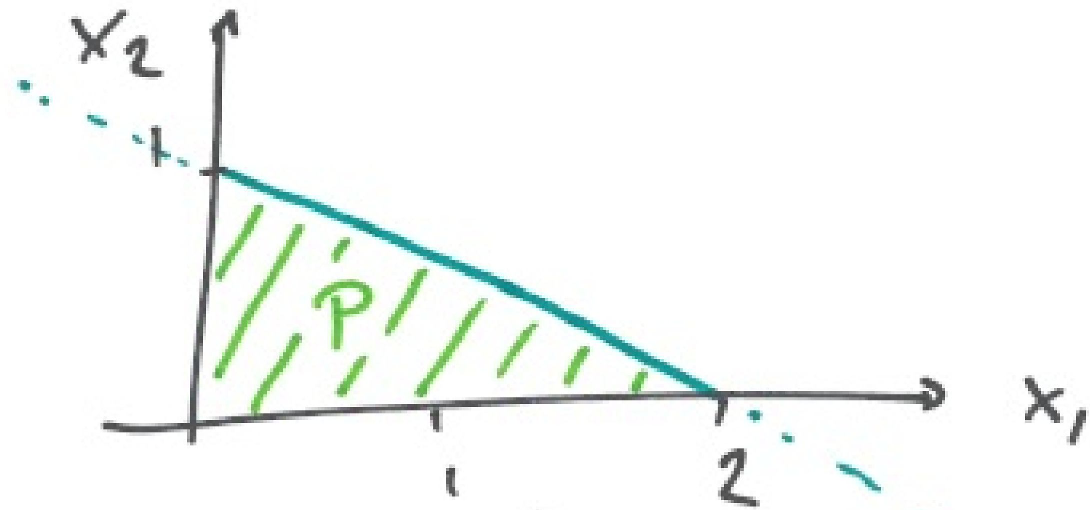
$A \in \mathbb{Z}^{m \times n}$ is TU, $b \in \mathbb{Z}^m$

$\Rightarrow$ $\{x \in \mathbb{R}^n : Ax \leq b\}$ has integer vertices

<u>Remark</u>: ($\Rightarrow$) sufficient but <u>not</u> necessary ~~($\Leftarrow$)~~

For example, take

$$P = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$



P has integral vertices $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

However $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$ is <u>not</u> TU.

Constructing TU matrices

Theorem 7. Let $A \in \mathbb{Z}^{m \times n}$ be a $-1, 0, +1$ matrix in which each column has

- at most one $+1$ entry, and
- at most one -1 entry.

Then A is TU.

Example

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & +1 & 0 & +1 & 0 \end{bmatrix}$$

Remark: $(\Rightarrow)$ sufficient but not necessary $\cancel{(\Leftarrow)}$

For example: $\left[\begin{array}{c|c} +1 & +1 \\ +1 & 0 \end{array}\right]$ is TU.

proof of Theorem 7:

We will show that every $k \times k$ submatrix M of A has $\det(M) \in \{-1, 0, +1\}$.

We proceed by induction on k.

Base case: $k=1$: true since entries are $-1, 0, +1$.

Suppose $k \geq 2$, and result holds for $k-1$.

Case 1: M has an all-zero column.

$$M = \left[\begin{array}{c|c} 0 & ? \end{array}\right], \quad \det(M) = 0 \quad \text{done.}$$

Case 2: every column has one +1 and one −1.

Let us sum all rows:

$$[1\ 1\ \cdots\ 1] \cdot \left[\begin{array}{c|c|c|c} +1 & -1 & +1 & \\ -1 & +1 & & -1 \\ & & -1 & \\ & & & +1 \end{array}\right] = [0\ 0 \cdots 0]$$

$$M^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ M^T not invertible

$\Rightarrow$ $\det(M^T) = \det(M) = 0$ done.

<u>Case 3:</u> There is a column of M with exactly one $+1$ or one -1.

$$M = \begin{bmatrix} e_i & \Big| & G \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} -e_i & \Big| & G \end{bmatrix}$$

G is $k \times (k-1)$, so every square submatrix H of G is $(k-1) \times (k-1)$ at most.

So by induction hypothesis,

H is unimodular (i.e. $\det(H) \in \{-1, 0, +1\}$

By definition, this means G is TU.

By Theorem 5(d), M is TU.

In particular, $\det(M) \in \{-1, 0, +1\}$.

$\longrightarrow$ done.

<u>Remark:</u> M is TU $\iff$ M^T is TU,

so Theorem 7 applies to rows

as well.

Application to flows

Let $G=(V,E)$ be a directed graph.

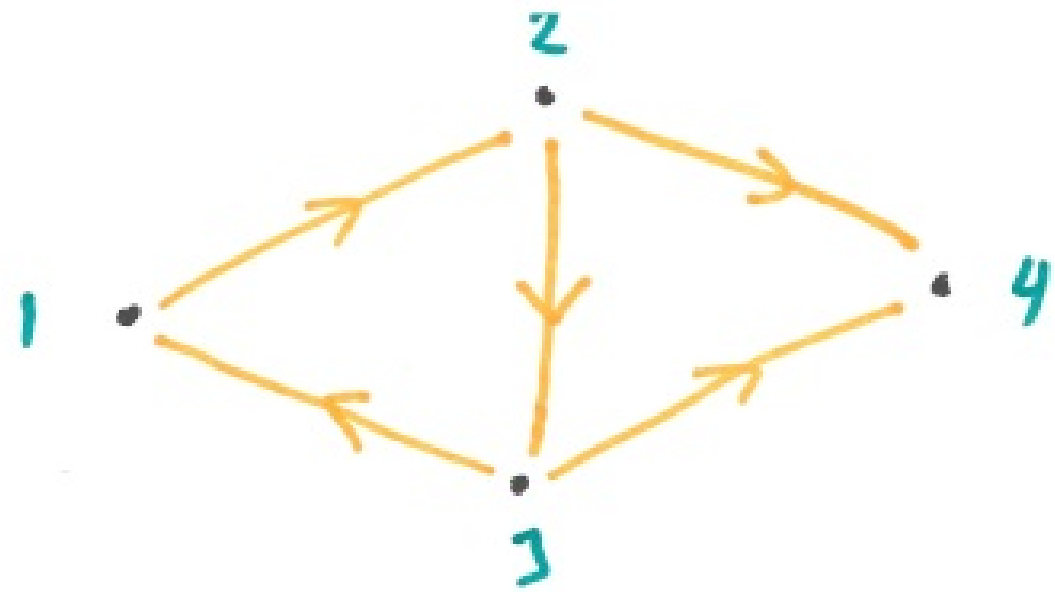
Definition The vertex-arc incidence matrix of G has rows indexed by V and columns indexed by E.

For every column/edge uv, we have $+1$ in row u,

-1 in row v,

0 everywhere else

Example:



$$M = \begin{array}{c c}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} &
\left[\begin{array}{c|c|c|c|c}
+1 & -1 & & & \\
-1 & & +1 & +1 & \\
 & +1 & -1 & & +1 \\
 & & & -1 & -1
\end{array}\right] \\
 & \begin{matrix} 12 & \;31 & \;23 & \;24 & \;34 \end{matrix}
\end{array}$$

Remark: For every row/vertex v, we have

$+1$ in every column e if $e \in \delta^+(v)$

-1 in every column e if $e \in \delta^-(v)$

0 otherwise

Therefore, a min-cost flow formulation for G

$$\begin{aligned} \max \quad & c^T x \\ & f_q(x) = b_q \quad , \quad \forall q \in V \\ & 0 \le x \le u \end{aligned}$$

where $$f_x(q) = \sum_{e \in \delta^+(q)} x_e - \sum_{e \in \delta^-(q)} x_e ,$$

can be rewritten

$$\max \; c^T x$$
$$Mx = b$$
$$0 \le x \le u.$$

In the example:

$$
\begin{array}{c}
\begin{bmatrix}
+1 & -1 & & & \\
-1 & & +1 & +1 & \\
& +1 & -1 & & +1 \\
& & & -1 & -1
\end{bmatrix} \\
\begin{array}{ccccc} 12 & 31 & 23 & 24 & 34 \end{array}
\end{array}
x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Theorem 8 Consider the min-cost flow formulation

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & f_x(q) = b_q \ , \quad \forall q \in V \\ & 0 \le x \le u \end{aligned} \qquad (P)$$

where $b \in \mathbb{Z}^{|V|}$, $u \in \mathbb{Z}^{|E|}$.

If (P) has an optimal solution, then it has an <u>integral</u> optimal solution.

<u>proof</u> Observe that (P) is equivalent to

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ u \\ 0 \end{bmatrix} \end{aligned}$$

where M is the vertex-arc incidence matrix of G.

By Theorem 7, M is TU.

By Theorem 5 (and its Corollary), $\begin{bmatrix} M \\ -M \\ I \\ -I \end{bmatrix}$ is TU.

By Theorem 6, the vertices of the feasible region of (P) are integral.

Since (P) is an LP, if it has an optimal solution, at least one is a vertex.

Theorem 9: Consider the max-flow problem

$$
(P) \qquad \begin{aligned} \max \quad & f_x(s) \\ \text{s.t.} \quad & f_x(q) = 0 \quad \forall q \in V \setminus \{s,t\} \\ & 0 \le x \le u. \end{aligned}
$$

If (P) has an optimal solution, then at least one is integral.

proof: Same as Theorem 8.