Math 115 Spring 2015: Assignment 9

Due: at the tutorial Thursday 7/23

Last name:

First name:

ID number:

Note: You need to show all the steps and the reasoning in obtaining your answers in order to receive full marks.

1. [5 marks] Let
$$\vec{v}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$
Knowing that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, write the vector $\vec{x} = \begin{bmatrix} 6 \\ -6 \\ 15 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

2. [5 marks] Let $S = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$, with
$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$
Find a basis of $S^\perp$. **Hint:** A vector is orthogonal to S if and only if it is orthogonal to both $\vec{u}_1$ and $\vec{u}_2$.

3. [10 marks] Let
$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$
Find an orthonormal basis of $\operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

4. [10 marks] Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix},$$

i.e. find an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

5. (a) [3 marks] Show that the following statement is false by providing a counter-example:

   Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$ and $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for some positive integer $k < n$. Then $S^\perp = \text{span}\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$.

   (b) [2 marks] What additional property would $\{\vec{v}_1, \ldots, \vec{v}_n\}$ need for the statement to be true? (no proof necessary)

6. [5 marks] Prove that the determinant of an orthogonal matrix is always 1 or -1. **Hint:** Recall that for any square matrix A, we have $\det(A) = \det(A^T)$. Also, if A is invertible, then $A^{-1}A = I$.