Math 115 Practice Final

Spring 2015

Exam details:

Course:	MATH 115 – Linear Algebra for Engineers
Instructor:	Laurent Poirrier
Duration of exam:	150 minutes (2h30)
Exam type:	Closed book – no additional materials are allowed

Instructions:

Your answers must be stated and justified in a clear and logical form, and you must show all of your steps in order to receive full marks. You may use any result from class without proof, unless you are being asked to prove this result. Simplify your answers as much as possible.

1. (a) Write a transformation matrix $A \in \mathbb{R}^{2\times 2}$ such that $(A \cdot \vec{x})$ is $\vec{x}$ rotated (counter-clockwise) by an angle of $-\frac{\pi}{4}$.

   (b) Write a matrix $B \in \mathbb{R}^{2\times 2}$ such that $(B \cdot \vec{x})$ is the vector $\vec{x}$ with both components scaled by a factor $25\sqrt{2}$.

   (c) Write a matrix $C \in \mathbb{R}^{2\times 2}$ such that $(C \cdot \vec{x})$ is the projection of $\vec{x}$ on $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   (d) Write a matrix $G \in \mathbb{R}^{2\times 2}$ such that $(G \cdot \vec{x})$ corresponds to the vector obtained by applying the operations described in (a) then (b) then (c) on $\vec{x}$ (in that order).

2. The matrix $A \in \mathbb{R}^{3\times 6}$ and its reduced row echelon form B are given as follows:

$$A = \begin{bmatrix} 1 & 2 & 1 & -2 & -1 & 1 \\ 2 & 4 & 1 & -3 & -3 & -2 \\ -1 & -2 & 0 & 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Determine the rank of A.

   (b) Determine a basis for the columnspace of A.

   (c) Determine a basis for the rowspace of A.

   (d) Determine the nullspace of A, i.e. the set $S = \{x \in \mathbb{R}^6 | A\vec{x} = \vec{0}\}$ of all solutions to the system $A\vec{x} = \vec{0}$. This set S is a subspace. Determine a basis for S.

3. Give all values of c for which the matrix $A = \begin{bmatrix} 1 & 3 \\ c & 2 \end{bmatrix}$ is invertible.

4. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$. Compute A^{-1}.

5. Let

$$A = \begin{bmatrix} 1 & z & z & z \\ 0 & 2 & x & x \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}.$$

Assume that $\det(A) = 5$.

(a) Give the numerical value of $\det(B)$, where

$$B = \begin{bmatrix} 1 & z & z & 2z \\ 0 & 2 & x & 2x \\ 0 & 0 & a & 2b \\ 0 & 0 & c & 2d \end{bmatrix}.$$

(b) Give the numerical value of $\det(C)$, where

$$C = \det \begin{bmatrix} 1 & z & z & z \\ 0 & 2 & x & x \\ 0 & 0+2 & a+x & b+x \\ 0 & 0 & c & d \end{bmatrix}.$$

6. Let

$$A = \begin{bmatrix} 1 & 6 & -3 \\ 0 & 4 & 0 \\ -3 & 6 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A. For each eigenvalue, give the algebraic multiplicity, determine a basis of the corresponding eigenspace, and give the geometric multiplicity.

(b) Is A diagonalizable? If so, diagonalize A by finding an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

7. Let $A = \begin{bmatrix} 1 & 2 \\ c & 2 \end{bmatrix}$.

(a) For what value of c does A have only one eigenvalue?

(b) If A has only one eigenvalue (i.e. if c takes the value found above), then compute this eigenvalue and give a basis of the corresponding eigenspace.

(c) Give all the values of c for which A is diagonalizable.

(d) Give all the values of c for which A is orthogonally diagonalizable.

8. (a) Prove that for any matrix $A \in \mathbb{R}^n$, the matrix $A^T A$ is symmetric.

(b) Prove that for any matrix $A \in \mathbb{R}^n$, the determinant of $A^T A$ is greater than or equal to 0.

(c) Give a counter-example to show that the following statement is false: "The determinant of a symmetric matrix is greater than or equal to 0".

(d) Use the answers above to prove that the following statement is false: "For any symmetric matrix $B \in \mathbb{R}^n$, there exists a matrix A such that $A^T A = B$".

9. Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$. Find an orthonormal basis of S.

10. Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$. Give a basis of $S^{\perp}$.

11. Prove that if A is orthogonally diagonalizable, then A^2 is orthogonally diagonalizable.

12. Find all the possible values for the complex number z in each of the following expressions. You may express your answers in standard form or polar form.

    (a) $z = \overline{\frac{(-1+2i)(-2-6i)}{i(-1+3i)}}$

    (b) $z = (\sqrt{3} + i)^4$

    (c) $z^4 = \sqrt{3} + i$

Common values of trigonometric functions

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$