Math 115 Practice Midterm

Spring 2015

Last name: ______________________________

First name: ______________________________

ID number: ______________________________

Exam details:

Course:	MATH 115 – Linear Algebra for Engineers
Instructor:	Laurent Poirrier
Duration of exam:	120 minutes
Exam type:	Closed book – no additional materials are allowed

Instructions:

Your answers must be stated and justified in a clear and logical form, and you must show all of your steps in order to receive full marks. You may use any result from class without proof, unless you are being asked to prove this result. Simplify your answers as much as possible.

1. [4 marks] Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Compute $\text{perp}_{\vec{w}}\, \vec{x}$.

2. The following sets are *not* subspaces. For each set, find a counter-example that proves that it is not a subspace (use your counter-example to show that it does not satisfy the definition of a subspace).

   (a) [2 marks] $R = \{x \in \mathbb{R}^2 \mid x_2 \leq -1\}$.

   (b) [2 marks] $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

   (c) [2 marks] $T = \{x \in \mathbb{R}^2 \mid x_1^2 = x_2\}$.

3. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$. Only one of the following statements is correct. For the one that is correct, just indicate that it is true (no justification necessary). For each of the other two, give a counter-example proving that it is false.

   (a) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3\}$.

   (b) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then at least one of (1), (2) and (3) is true:
   (1) $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3\}$,
   (2) $\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_3\}$,
   (3) $\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$.

   (c) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then at most one of (1), (2) and (3) is true:
   (1) $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3\}$,
   (2) $\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_3\}$,
   (3) $\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$.

4. [2 marks] Write a transformation matrix $H \in \mathbb{R}^{2\times 2}$ such that $(H \cdot \vec{x})$ is $\vec{x}$ rotated (counter-clockwise) by an angle of $\frac{\pi}{4}$. Note: $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

5. (a) [1 mark] Write a matrix $A \in \mathbb{R}^{2\times 2}$ such that if $\vec{y} = A \cdot \vec{x}$, then $\vec{y}$ corresponds to a shear of $\vec{x}$ with shear factor s (i.e. $y_1 = x_1 + sx_2$).

   (b) [1 mark] Write a matrix $B \in \mathbb{R}^{2\times 2}$ such that if $\vec{z} = B \cdot \vec{y}$, then $\vec{z}$ corresponds to scaling the components of $\vec{y}$ by factors a and b (i.e. $z_1 = ay_1$ and $z_2 = by_2$).

   (c) [2 marks] Write a matrix $G \in \mathbb{R}^{2\times 2}$ such that if $\vec{z} = G\vec{x}$, then $\vec{z}$ corresponds to first shearing $\vec{x}$ with a shear factor s, then scaling the result by factors a and b. Note that this corresponds to applying the transformation represented by the matrix A, then the one represented by B. The variables s, a and b may appear in G.

   (d) [4 marks] Find values of s, a and b such that $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = G \cdot \begin{bmatrix} -5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = G \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

6. Let $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.

   (a) [2 marks] Write a matrix whose columnspace is $\text{span}\{v_1, v_2, v_3, v_4\}$.

(b) [2 marks] Write a matrix whose rowspace is $\text{span}\{v_1, v_2, v_3, v_4\}$.

7. The matrix $A \in \mathbb{R}^{4\times 7}$ and its reduced row echelon form B are given as follows:

$$A = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 1 & -5 \\ 0 & 0 & -1 & 5 & -9 & -1 & 4 \\ 3 & -9 & -1 & 8 & 3 & 1 & -5 \\ -1 & 3 & 1 & -6 & 5 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & -5 & 9 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [2 marks] Determine the rank of A.

(b) [2 marks] Determine a basis for the columnspace of A.

(c) [2 marks] Determine a basis for the rowspace of A.

(d) [5 marks] The set $S = \{x \in \mathbb{R}^7 | A\vec{x} = \vec{0}\}$ is the set of all solutions to the system $A\vec{x} = \vec{0}$. This set S is a subspace. Determine a basis for S.

8. [5 marks] Prove that if $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices and $AB = BA$, then (AB) is a symmetric matrix.

9. [8 marks] Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
Prove that if $\vec{x} \cdot \vec{w} \neq 0$ for all $\vec{w} \in \text{span}\{\vec{e}_3\}$, then $\vec{x} \notin \text{span}\{\vec{e}_1, \vec{e}_2\}$.