Math 115 Spring 2015: Quiz 1

Solutions

1. [3 marks] Let $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Find a vector $\vec{x}$ of length 20 such that $\vec{x}$ is on the line $\vec{0} + t\vec{v}$, $t \in \mathbb{R}$.

   Solution: Any vector on the line $\vec{0} + t\vec{v}$ takes the form $\vec{y} = t\vec{v}$, for some value of $t \in \mathbb{R}$. We are looking for $\vec{y}$ such that $||\vec{y}|| = 20$, i.e. $||\vec{y}|| = ||t\vec{v}|| = |t|.||\vec{v}|| = 20$, so $|t| = \frac{20}{||\vec{v}||}$.

   $$||\vec{v}|| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

   so $|t| = \frac{20}{5} = 4$. A solution is given by

   $$\vec{y} = 4 \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 12 \\ -16 \end{bmatrix}.$$

2. [3 marks] Determine a scalar equation (i.e. the values of a, b, c, d in $ax_1 + bx_2 + cx_3 = d$) of the plane in $\mathbb{R}^3$ whose normal vector is $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ and contains the point $(2, 2, -1)$.

   Solution: Because a normal vector to the plane is given by

   $$\vec{n} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix},$$

   we know that the equation of the plane takes the form

   $$\vec{n}.\vec{x} = -2x_1 + 3x_2 + x_3 = d$$

   for some value of d. We now find the value of d by exploiting the fact that the point $(2, 2, -1)$ belongs to the plane, so $-2x_1+3x_2+x_3 = d$ for $x_1 = 2$, $x_2 = 2$ and $x_3 = -1$. This gives $-2.2+3.2+(-1) = -4+6-1 = 1 = d$. An equation of the plane is therefore given by

   $$-2x_1 + 3x_2 + x_3 = 1.$$

3. [3 marks] Determine the point on the line $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ that satisfies $x_1 - 2x_2 + x_3 = 4$.

   Solution: The point needs to satisfy

   $$\begin{cases} x_1 = 1 + t \\ x_2 = -1 + 2t \\ x_3 = -1 - t \\ x_1 - 2x_2 + x_3 = 4 \end{cases}.$$

This yields $(1+t)+2-4t-1-t=4$, so $-4t=4-1-2+1=2$, and finally $t=-\frac{1}{2}$. We obtain

$$\vec{x}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\-2\\-\frac{1}{2}\end{bmatrix}.$$

4. [3 marks] Let $\vec{u},\vec{v}\in\mathbb{R}^n$ be unit vectors (i.e. $||\vec{u}||=||\vec{v}||=1$) such that $||\vec{v}-\vec{u}||=1$. Prove that $\vec{u}.\vec{v}=K$ where $K\in\mathbb{R}$ is a constant real number, and determine the numeric value of K. (Hint: Expand and simplify the expression $||\vec{v}-\vec{u}||^2=1^2$.)

Solution: Expanding $||\vec{v}-\vec{u}||^2=1^2$ gives

$$\begin{aligned}||\vec{v}-\vec{u}||^2 &= 1\\(\vec{v}-\vec{u})(\vec{v}-\vec{u}) &= 1\\\vec{v}\vec{v}-\vec{u}\vec{v}-\vec{v}\vec{u}+\vec{u}\vec{u} &= 1\\||\vec{v}||^2-2\vec{u}\vec{v}+||\vec{u}\vec{u}||^2 &= 1\\1-2\vec{u}\vec{v}+1 &= 1\\\vec{u}\vec{v} &= \frac{1}{2}=K.\end{aligned}$$

5. [3 marks] Let $\vec{u}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\vec{v}=\begin{bmatrix}4\\2\\4\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}2\\1\\1\end{bmatrix}$. Determine the set of all vectors $\vec{x}\in\mathbb{R}^3$ such that: (a) $\vec{x}$ is a linear combination of $\vec{u}$ and $\vec{v}$, and (b) $\vec{x}$ is orthogonal to $\vec{w}$. Express your set of vectors as either a single point, a vector equation of a line, or a scalar equation of a plane.

Solution: By (a), $\vec{x}$ can be written $\vec{x}=s\vec{u}+t\vec{v}$ for some $s,t\in\mathbb{R}$. By (b), $\vec{x}.\vec{w}=0$. Writing a system combining both, we obtain

$$\begin{cases}\vec{x}=s\begin{bmatrix}1\\1\\-1\end{bmatrix}+t\begin{bmatrix}4\\2\\4\end{bmatrix}\\\vec{x}.\begin{bmatrix}2\\1\\1\end{bmatrix}=0\end{cases},$$

which can be rewritten

$$\begin{cases}x_1=s+4t\\x_2=s+2t\\x_3=-s+4t\\2x_1+x_2+x_3=0\end{cases}.$$

Using the expressions of x_1, x_2, x_3 given by the first three equations in the fourth, we obtain

$$\begin{aligned} 2(s+4t)+(s+2t)+(-s+4t) &= 0 \\ 2s+8t+s+2t-s+4t &= 0 \\ 2s+14t &= 0 \\ s &= -7t. \end{aligned}$$

Now, we replace s by $-7t$ in the first three equations, yielding

$$\begin{cases} x_1 = -7t + 4t = -3t \\ x_2 = -7t + 2t = -5t \\ x_3 = 7t + 4t = 11t \end{cases},$$

or equivalently, the vector equation of a line

$$\vec{x} = \vec{0} + t \begin{bmatrix} -3 \\ -5 \\ 11 \end{bmatrix}.$$