Math 115 Spring 2015: Quiz 2

Solutions

1. [5 marks] Let $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{R}^n$. Prove that if $\{\vec{u}_1, \vec{u}_2\}$ is linearly dependent, then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is also linearly dependent.

   Solution: Since $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent, there exist $s, t \in \mathbb{R}$ not both zero such that $s\vec{v}_1 + t\vec{v}_2 = \vec{0}$. Therefore, we can write $s\vec{v}_1 + t\vec{v}_2 + 0\vec{v}_3 = \vec{0}$, where either s or t is nonzero. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

2. Let $T = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ -5 \\ 4 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}.$$

   (a) [5 marks] Prove that $\vec{v}_4 \in \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

   (b) [5 marks] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

   (c) [5 marks] Find a basis of T. Prove that the vectors in the basis are linearly independent.

Solution:

(a) Solve the system $\vec{v}_4 = r\vec{v}_1 + s\vec{v}_3 + t\vec{v}_2$.

$$\begin{cases} 5 = r - s - 3t \\ 3 = 3r + s - 5t \\ -4 = -2r + 4t \end{cases} \rightarrow \begin{cases} s = r - 3t - 5 \\ 3 = 3r + r - 3t - 5 - 5t \\ r = 2 + 2t \end{cases} \rightarrow \begin{cases} s = 2 + 2t - 3t - 5 \\ 4r - 8t = 8 \\ r = 2 + 2t \end{cases} \rightarrow \begin{cases} s = -t - 3 \\ 0 = 0 \\ r = 2 + 2t \end{cases}$$

Thus, for any value of t, letting $s = -t - 3$ and $r = 2t + 2$ solves the system. For example, we can set $t = 1$, and obtain that $\vec{v}_4 = 4\vec{v}_1 - 4\vec{v}_3 + \vec{v}_2$.

Alternatively, one could directly notice that $\vec{v}_4 = 2\vec{v}_1 - 3\vec{v}_3$.

(b) Solve the system $r\vec{v}_1 + s\vec{v}_3 + t\vec{v}_2 = \vec{0}$.

$$\begin{cases} r - s - 3t = 0 \\ 3r + s - 5t = 0 \\ -2r + 4t = 0 \end{cases} \rightarrow \begin{cases} s = r - 3t \\ 3r + r - 3t - 5t = 0 \\ r = 2t \end{cases} \rightarrow \begin{cases} s = 2t - 3t \\ 4t - 8t = 0 \\ r = 2t \end{cases} \rightarrow \begin{cases} s = -t \\ 0 = 0 \\ r = 2t \end{cases}$$

Thus, for any value of t, letting $s = -t$ and $r = 2t$ solves the system. For example, we can set $t = 1$, and obtain that $2\vec{v}_1 - \vec{v}_3 + \vec{v}_2 = \vec{0}$.

Alternatively, one could directly notice that $\vec{v}_2 = -2\vec{v}_1 + \vec{v}_3$

(c) We know from the above that $\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_3\}$. We now check whether $\{\vec{v}_1, \vec{v}_3\}$ is linearly independent. We solve $s\vec{v}_1 + t\vec{v}_3 = \vec{0}$.

$$\begin{cases} r - s = 0 \\ 3r + s = 0 \\ -2r = 0 \end{cases} \rightarrow \begin{cases} s = r \\ s = -3r \\ r = 0 \end{cases}$$

The only solution is $s = r = 0$, so $\{\vec{v}_1, \vec{v}_3\}$ is linearly independent. The set $\{\vec{v}_1, \vec{v}_3\}$ is thus a basis of T.

Note: Only one basis of T is asked, but all the following are bases of T: $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_4\}$, $\{\vec{v}_3, \vec{v}_2\}$, $\{\vec{v}_3, \vec{v}_4\}$, $\{\vec{v}_2, \vec{v}_4\}$. There are also many other possibilities, like for example

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$