Math 115 Spring 2015: Quiz 3

Solutions

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. We consider the system $r\vec{u} + s\vec{v} + t\vec{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$.

   (a) [2 marks] Write the augmented matrix of the system.
   Solution:
   $$\left[\begin{array}{rrr|r} 2 & 1 & 0 & 3 \\ 1 & -2 & 1 & 4 \\ 2 & 1 & 0 & 3 \end{array}\right]$$

   (b) [6 marks] Compute the reduced row echelon form (RREF) of that matrix.
   Solution:
   $$\sim \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & -\frac{5}{2} & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R1' = \frac{1}{2}R1 \\ R2' = R2 - \frac{1}{2}R1 \\ R3' = R3 - R1 \end{array}$$
   $$\sim \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{5} & 2 \\ 0 & 1 & -\frac{2}{5} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R1'' = R1' + \frac{1}{5}R2' \\ R2'' = -\frac{2}{5}R2' \\ R3'' = R3' \end{array}$$

   (c) [4 marks] Give all the solutions to the system (the general solution).
   Solution:
   $$\begin{cases} x_1 = 2 - \frac{1}{5}t \\ x_2 = -1 + \frac{2}{5}t \\ x_3 = t \end{cases}, \text{ for all } t \in \mathbb{R}$$

2. A system has the following augmented matrix in reduced row echelon form (RREF):
   $$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

   (a) [2 marks] How many solutions does it have (zero, exactly one, or infinitely many)?
   Solution: It has infinitely many solutions, because it has fewer pivots than variables (2 pivots and 4 variables).

   (b) [2 marks] How many parameters are there in the general solution (i.e. how many free variables)?
   Solution: The number of parameters in the general solution is the number of variables minus the number of pivots in the RREF (the rank of the matrix), so it is $4 - 2 = 2$.

   (c) [4 marks] Give a matrix that corresponds to the same system, but which is not in row echelon form (REF).
   Solution: We can, for example, add the first row to the second, obtaining
   $$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 5 & 2 \\ 1 & 2 & 1 & 7 & 3 \end{array}\right].$$

The operation is allowed, so the result corresponds to an equivalent system. However, this matrix is not in REF, because the leading elements in both rows are in the same column.