# Math 115 Spring 2015: Quiz 5

## Solutions

1. [4 marks] Let $E = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. For a given vector $\vec{x} \in \mathbb{R}^2$, we compute $\vec{y} = E\vec{x}$. Then we compute $\vec{z} = F\vec{y}$. Find a single matrix $G \in \mathbb{R}^{2\times 2}$ such that $\vec{z} = G\vec{x}$.

   **Solution:** We know that $\vec{z} = F\vec{y} = F(E\vec{x}) = FE\vec{x}$, so $G = FE$:

$$G = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 3 & 2 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \end{bmatrix}.$$

   (a) [4 marks] Compute the RREF of $A$.

   **Solution:** We perform the following two row operations: $R2' = R2 - R1$ and $R3' = R3 - R1$.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

   (b) [1 marks] Determine the rank of $A$.

   **Solution:** There are three leading ones in the RREF of $A$, so $\operatorname{rank}(A) = 3$.

   (c) [3 marks] Determine a basis of the columnspace of $A$.

   **Solution:** One basis of $\operatorname{col}(A)$ is given by the columns of $A$ that correspond to columns of its RREF having a leading one. Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   is a basis of $\operatorname{col}(A)$.

   (d) [3 marks] Determine a basis of the rowspace of $A$.

**Solution:** One basis of row$(A)$ is given by the nonzero rows of $B$. Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis of row$(A)$.

(e) [5 marks] Let $S$ be the set of solutions to the system $A \cdot \vec{x} = \vec{0}$. $S$ is a subspace. Determine a basis of $S$.

**Solution:** We write the RREF of the system $A\vec{x} = \vec{0}$. Because the right-hand side is zero in the system, it will be zero in its RREF too. The other coefficients in the RREF will simply be the coefficients in the RREF of $A$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

The general solution to this system is

$$\begin{cases} x_1 &= -2x_3 & -1x_5 \\ x_2 &= -2x_3 & +1x_5 \\ x_4 &= & 1x_5 \end{cases},$$

where $x_3$ and $x_5$ are free variables. As a vector equation, we can write this subspace as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t \quad \text{for all } s, t \in \mathbb{R}.$$

It is easy to verify that the only solution for $\vec{x} = 0$ is $s = t = 0$, so the two vectors above are linearly independent. Therefore,

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis of $S$.