Math 115 Spring 2015: Quiz 7

Solutions

1. [5 marks] Find the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 3 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

Hint: Use row operations to put the matrix in REF or RREF.

Solution: If we multiply the first row of A by $\frac{1}{2}$, we obtain the matrix

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 3 \\ 4 & 4 & 4 & 0 \end{bmatrix}$$

whose determinant is half of the determinant of A. Then, subtracting multiples of the first row to the others, we obtain

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 3 \\ 4 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} = A''$$

Because A'' is triangular, its determinant is the product of its diagonal entries, so $\det(A'') = 1.1.1.(-4) = -4 = \det(A')$. Since $\det(A') = \frac{1}{2}\det(A)$, we have $\det(A) = -8$.

2. [5 marks] The determinant of the following matrix is a constant number independent of $p, q, c \in \mathbb{R}$. Find that number.

$$B = \begin{bmatrix} p & q & c \\ p^2 - p & q(p-1) & pc - c \\ 0 & 0 & 1 \end{bmatrix}$$

Hint: Use row operations to put the matrix in REF.

Solution: Observe that

$$B = \begin{bmatrix} p & q & c \\ p(p-1) & q(p-1) & c(p-1) \\ 0 & 0 & 1 \end{bmatrix}.$$

We subtract $(p-1)$ times the first row from the second, and obtain

$$\begin{bmatrix} p & q & c \\ p(p-1) & q(p-1) & c(p-1) \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} p & q & c \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The latter matrix has a row of zeros, so its determinant is zero. So $\det(B) = 0$.

3. [5 marks] Let C be a 4×4 matrix whose determinant is -32. What is the determinant of $\frac{1}{2}C$?

Solution: The matrix $\frac{1}{2}C$ can be obtained by performing 4 elementary row operations on C. Each multiplies one row of C by $\frac{1}{2}$. Therefore, $\det(\frac{1}{2}C) = \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\det(C) = \frac{1}{16}(-32) = -2$.

Alternative solution:

$$\det(\frac{1}{2}C) = \det\left(\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot C\right) = \det\left(\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}\right) \cdot \det(C) = \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}(-32) = -2.$$

4. [5 marks] The vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. Find the corresponding eigenvalue.

Solution:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so the corresponding eigenvalue is 3.