Math 115 Spring 2015: Quiz 8

Solutions

1. [10 marks] Let $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$. Compute the eigenvalues of A. For each eigenvalue, give a basis of the corresponding eigenspace.

Solution: First, we compute the eigenvalues of A as the roots of its characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = (-2-\lambda)\cdot(2-\lambda) = (\lambda+2)\cdot(\lambda-2).$$

The two eigenvalues are thus -2 and 2.

First eigenvalue: $\lambda = -2$.

Corresponding eigenvectors $\vec{u}$ are solutions to the system $A\vec{u} = -2\vec{u}$, which is equivalent to $(A+2I)\vec{u} = \vec{0}$. The augmented matrix of this system is

$$\left[\begin{array}{cc|c} -2+2 & 3 & 0 \\ 0 & 2+2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We thus have $0u_1 + 1u_2 = 0$, so $u_2 = 0$,

$$\vec{u} = s \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s \in \mathbb{R}$$

and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis of the eigenspace associated to the eigenvalue -2.

Second eigenvalue: $\lambda = 2$.

Corresponding eigenvectors $\vec{v}$ are solutions to the system $A\vec{v} = 2\vec{v}$, which is equivalent to $(A-2I)\vec{v} = \vec{0}$. The augmented matrix of this system is

$$\left[\begin{array}{cc|c} -2-2 & 3 & 0 \\ 0 & 2-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We thus have $v_1 = \frac{3}{4}v_2$, so

$$\vec{v} = t \cdot \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

and $\left\{ \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace associated to the eigenvalue 2.

2. [5 marks] Let

$$G = \begin{bmatrix} -1 & 0 & -3 & 6 \\ -\frac{7}{4} & \frac{5}{2} & -\frac{1}{4} & 2 \\ -\frac{1}{2} & 1 & \frac{9}{2} & -2 \\ -\frac{1}{4} & \frac{1}{2} & \frac{5}{4} & 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of G are

- 2 (with corresponding eigenspace $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$),
- 4 (with corresponding eigenspace $\operatorname{span}\{\vec{v}_3\}$), and
- -1 (with corresponding eigenspace $\operatorname{span}\{\vec{v}_4\}$).

Find an invertible matrix P and a diagonal matrix D such that the matrix equation $P^{-1}GP = D$ is satisfied.
Note: It is not necessary to compute P^{-1} or verify that the equation is satisfied. Giving P and D is enough.

Solution: We let P be the matrix constructed by gathering basis vectors for the eigenspaces of G:

$$P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4] = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and D be a diagonal matrix whose diagonal elements are the corresponding eigenvalues of G:

$$D = \operatorname{diag}(2, 2, 4, -1) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We have seen in class that under such conditions, $P^{-1}GP = D$.

3. [5 marks] Let $B \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix such that $P^{-1}BP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of B. Assuming that B is invertible, give an expression of B^{-1} in terms of P, P^{-1} and $\lambda_1, \ldots, \lambda_n$.

Solution: Pre-multiply both sides of

$$P^{-1}BP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

by P and post-multiply them by P^{-1}. We obtain

$$B = P \operatorname{diag}(\lambda_1, \ldots, \lambda_n) P^{-1}.$$

Since $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, its inverse is

$$(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))^{-1} = \operatorname{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n})$$

so we get

$$B = P\,\mathrm{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n})P^{-1}.$$