Math 115 Spring 2015: Assignment 1

Solutions

Notice: The solution to question 1c uses material that will be covered in class on Monday 5/11. Answering it correctly will give you bonus marks.

1. Consider the two vectors $\vec{u} = \begin{bmatrix} -2 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -4 \\ 2 \\ 5 \end{bmatrix}$.

   (a) [3 marks] Determine a vector of length 3 that has the same direction as $\vec{u}$.

   Solution: Any vector that has the same direction as $\vec{u}$ takes the form $t\vec{u}$ for some $t \geq 0$. Let t be such that $t\vec{u}$ is the vector we are looking for. Then, $||t\vec{u}|| = 3$. Note that $||t\vec{u}|| = t||\vec{u}||$ so $t = \frac{3}{||\vec{u}||}$. We compute $||\vec{u}|| = \sqrt{(-2)^2 + 4^2 + 4^2 + 0^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$, yielding $t = \frac{3}{6}$. Thus, the vector

   $$\frac{1}{2}\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

   answers the question.

   (b) [3 marks] Determine the angle between $\vec{u}$ and $\vec{v}$.

   Solution: As we have seen, if θ is the angle between $\vec{u}$ and $\vec{v}$, then $\cos\theta = \frac{\vec{u}.\vec{v}}{||\vec{u}||\,||\vec{v}||}$, giving

   $$\begin{aligned} \theta &= \arccos\left(\frac{(-2).2 + 4.(-4) + 4.2 + 0.5}{6.\sqrt{2^2 + (-4)^2 + 2^2 + 5^2}}\right) \\ &= \arccos\left(\frac{-4 - 16 + 8}{6.\sqrt{4 + 16 + 4 + 25}}\right) \\ &= \arccos\left(\frac{-12}{6.\sqrt{49}}\right) = \arccos\left(\frac{-12}{6.7}\right) = \arccos\left(-\frac{2}{7}\right) \end{aligned}$$

   (c) [4 bonus marks] Write $\vec{v}$ as the sum of two nonzero orthogonal vectors, one of which is a scalar multiple of $\vec{u}$.

Solution: We may write $\vec{v} = \operatorname{proj}_{\vec{u}} \vec{v} + \operatorname{perp}_{\vec{u}} \vec{v}$, where

$$\begin{aligned}
\operatorname*{proj}_{\vec{u}} \vec{v} &= \vec{u}\frac{\vec{u}.\vec{v}}{||\vec{u}||^2} \\
&= \vec{u}\frac{-12}{6^2} = \vec{u}\frac{-12}{36} = \vec{u}\frac{-1}{3} \\
&= \begin{bmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ -\frac{4}{3} \\ 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname*{perp}_{\vec{u}} \vec{v} &= \vec{v} - \operatorname*{proj}_{\vec{u}} \vec{v} \\
&= \begin{bmatrix} 2-\frac{2}{3} \\ -4+\frac{4}{3} \\ 2+\frac{4}{3} \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{8}{3} \\ \frac{10}{3} \\ 5 \end{bmatrix}.
\end{aligned}$$

2. [3 marks] Let $\vec{u} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Determine the set of all vectors $\vec{x}$ such that $||\vec{u} - \vec{x}|| = ||\vec{v} - \vec{x}||$ (i.e. the distance between $\vec{x}$ and $\vec{u}$ is the same as the distance between $\vec{x}$ and $\vec{v}$).
   Solution: Note that $||\vec{u} - \vec{x}|| = ||\vec{v} - \vec{x}||$ is equivalent to $||\vec{u} - \vec{x}||^2 = ||\vec{v} - \vec{x}||^2$. We just write down the latter explicitly as
   $$\left|\left|\begin{bmatrix} 5-x_1 \\ -3-x_2 \end{bmatrix}\right|\right|^2 = \left|\left|\begin{bmatrix} 1-x_1 \\ 5-x_2 \end{bmatrix}\right|\right|^2$$
   to obtain $(5-x_1)^2 + (-3-x_2)^2 = (1-x_1)^2 + (5-x_2)^2$. We develop to get
   $$25 + x_1^2 - 10x_1 + 9 + x_2^2 + 6x_2 = 1 + x_1^2 - 2x_1 + 25 + x_2^2 - 10x_2$$
   and finally $-8x_1 + 16x_2 = -8$. We can multiply both sides by $-\frac{1}{4}$, yielding $x_1 - 2x_2 = 1$. This is the equation of a line in $\mathbb{R}^2$. The vectors we are looking form take the form $\{\vec{x} \in \mathbb{R}^2 : x_1 - 2x_2 = 1\}$.

   Optionally, we can express the set of all $\vec{x}$ parametrically. The line passes through $(1, 0)$ because $1 - 2.0 = 1$, and it has the direction $(2, 1)$. Thus, all vectors $\vec{x}$ take the form
   $$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{for } t \in \mathbb{R}.$$

3. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

   Proposition 1. *[3 marks] Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then, $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$.*

   Proposition 2. *[4 marks] Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then, $||\vec{u} + \vec{v}||^2 + ||\vec{u} - \vec{v}||^2 = 2||\vec{u}||^2 + 2||\vec{v}||^2$.*

Solution: For Proposition 1, the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

provide a counter-example. Indeed, $||\vec{u}+\vec{v}||^2 = 2^2 + 1^2 = 5$ and $||\vec{u}||^2 + ||\vec{v}||^2 = 1^2 + 0^2 + 1^2 + 1^2 = 3$.

Proposition 2 is true. Indeed,

$$\begin{aligned} ||\vec{u}+\vec{v}||^2 + ||\vec{u}-\vec{v}||^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) + (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) \\ &= (\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}) + (\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}) \\ &= 2\vec{u}\cdot\vec{u} + 2\vec{v}\cdot\vec{v} \\ &= 2||\vec{u}||^2 + 2||\vec{v}||^2. \end{aligned}$$