Math 115 Spring 2015: Assignment 2

Solutions

1. Determine whether or not the following sets are subspaces in their respective vector spaces. If so, prove it using the definition of subspaces. If not, provide a counterexample where a rule of subspaces is violated.

   (a) [2 marks] $S = \{\vec{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 = 0 \text{ and } x_1 - 3x_2 = 1\}$.

   Solution: S is not a subspace. For example, $\begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix} \in S$, but $0. \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$ because $0 - 3.0 = 0 \neq 1$. (Note that $\begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$ is the only vector that satisfies both equations of S, i.e. $S = \left\{ \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix} \right\}$).

   (b) [2 marks] $T = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{v} \geq 0\}$, where $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$.

   Solution: T is not a subspace. Let $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. The product $\vec{x}.\vec{v} = 2.2 + 2.2 + (-1).(-1) = 9 \geq 0$ so $\vec{x} \in T$. However, $((-1).\vec{x}).\vec{v} = (-2).2 + (-2).2 + (1).(-1) = -9 \ngeq 0$, so $((-1).\vec{x}) \notin T$. Therefore, the set is not closed under scalar multiplication.

2. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -3 \\ -2 \\ -3 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

   (a) [3 marks] Show that $\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$ is linearly dependent.
   Solution: We solve the system $p\vec{u} + q\vec{v} + r\vec{w} + s\vec{x} = \vec{0}$, i.e.

$$\begin{cases} p + 2q - 3r = 0 \\ 2p + 2q - 2r + 2s = 0 \\ -p + q - 3r - 3s = 0 \end{cases} \rightarrow \begin{cases} p = -2q + 3r \\ -4q + 6r + 2q - 2r + 2s = 0 \\ 2q - 3r + q - 3r - 3s = 0 \end{cases} \rightarrow \begin{cases} p = -2q + 3r \\ -2q + 4r + 2s = 0 \\ 3q - 6r - 3s = 0 \end{cases}$$

$$\rightarrow \begin{cases} p = -2q + 3r \\ s = q - 2r \\ 3q - 6r - 3q + 6r = 0 \end{cases} \rightarrow \begin{cases} p = -2q + 3r \\ s = q - 2r \\ 0 = 0. \end{cases}$$

   If we pick any value for q and r, and then compute p and s according to the above equation, we obtain a solution to the system. Note that we do not pick $q = r = 0$, because then we also obtain $p = s = 0$, which does not provide the “not all zero” solution necessary to prove linear dependence.

   For example, we can choose $q = 0, r = -1$, yielding $p = -3, s = 2$, and verify that

$$-3\vec{u} - \vec{w} + 2\vec{x} = 0.$$

Alternatively, one could notice e.g. that $\vec{w} = \vec{u} - 2\vec{v}$ (thus $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$), or that $\vec{x} = 2\vec{u} - \vec{v}$ (thus $\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$). Then, the theorem from the course implies linear dependence.

(b) [4 marks] Find a basis for $\text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$, i.e. a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ such that (a) $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, and (b) $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$. (Note: k will be smaller than 4, so it could be 1, 2 or 3.)

Solution: We proved above that $-3\vec{u} - \vec{w} + 2\vec{x} = 0$. Therefore $\vec{w} = -3\vec{u} + 0\vec{v} + 2\vec{x}$, thus $\vec{w} \in \text{span}\{\vec{u}, \vec{v}, \vec{x}\}$, so $\text{span}\{\vec{u}, \vec{v}, \vec{x}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$. Let us check that $\{\vec{u}, \vec{v}, \vec{x}\}$ is linearly independent by solving the system $p\vec{u} + q\vec{v} + s\vec{x} = \vec{0}$, i.e.

$$\begin{cases} p + 2q = 0 \\ 2p + 2q + 2s = 0 \\ -p + q - 3s = 0 \end{cases} \to \begin{cases} p = -2q \\ -4q + 2q + 2s = 0 \\ 2q + q - 3s = 0 \end{cases} \to \begin{cases} p = -2q \\ s = q \\ 0 = 0 \end{cases}$$

Again, for any value of q, we find a solution to the system. In particular, if $q = -1$, then $s = -1, p = 2$, showing that $2\vec{u} - \vec{v} - \vec{x} = \vec{0}$, i.e. the system is linearly dependent. Rearranging the previous equation, we see that $\vec{x} = 2\vec{u} - \vec{v}$, so $\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$, implying that $\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{x}\}$.

Let us now check that $\{\vec{u}, \vec{v}\}$ is linearly independent. The system $p\vec{u} + q\vec{v} = \vec{0}$, i.e.

$$\begin{cases} p + 2q = 0 \\ 2p + 2q = 0 \\ -p + q = 0 \end{cases} \to \begin{cases} p = -2q \\ p = q \\ p = q \end{cases}$$

has a solution only if $-2q = q$. This happens only if $p = q = 0$. Therefore, $\{\vec{u}, \vec{v}\}$ is linearly independent. To summarize, we showed that so $\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$ and $\{\vec{u}, \vec{v}\}$ is linearly independent. Thus, $\{\vec{u}, \vec{v}\}$ is a basis of $\text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$.

3. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

(a) [3 marks] Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent, then $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$.

Solution: False. For example, take $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Clearly, $0\vec{u} + \vec{v} - \frac{1}{2}\vec{w} = \vec{0}$, so they are linearly dependent. But $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

Note: If $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly dependent, then we know that *at least one* of the following statements is true:

(1) $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$.

(2) $\vec{v} \in \text{span}\{\vec{u}, \vec{w}\}$.

(3) $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$.

However, they are not necessarily *all* true. In the above example, (2) and (3) hold true, but (1) does not. As a consequence, $\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$, but $\text{span}\{\vec{v}, \vec{w}\} \neq \text{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Indeed $\text{span}\{\vec{v}, \vec{w}\}$ is the line $\left\{\vec{x} \in \mathbb{R}^2 \mid \vec{x} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}\right\}$, while $\text{span}\{\vec{u}, \vec{v}, \vec{w}\}$ is the entire plane $\mathbb{R}^2$.

(b) [3 marks] Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be two nonzero vectors (i.e. $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$). If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\{\vec{u}, \vec{v}\}$ is linearly independent.

Solution: True. We prove that if $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ were such that $\{\vec{u}, \vec{v}\}$ is linearly dependent, then $\vec{u}$ and $\vec{v}$ could not be orthogonal.

Assume $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then, there exist s, t not both zero such that $s\vec{u} + t\vec{v} = \vec{0}$. Furthermore, we know that both $s \neq 0$ and $t \neq 0$. Indeed, if $s = 0$ and $t \neq 0$, then $t\vec{v} = \vec{0}$ which contradicts $\vec{v} \neq \vec{0}$, and if $s \neq 0$ and $t = 0$, then $s\vec{u} = \vec{0}$ which contradicts $\vec{u} \neq \vec{0}$. Therefore, we can write $\vec{u} = -\frac{t}{s}\vec{v}$. So the product $\vec{u}.\vec{v} = -\frac{t}{s}\vec{v}.\vec{v} = -\frac{t}{s}||\vec{v}||^2 \neq 0$, showing that the vectors are not orthogonal.

(c) [3 marks] Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be three nonzero vectors. If (a) $\vec{u}$ is orthogonal to $\vec{v}$, and (b) $\vec{u}$ is orthogonal to $\vec{w}$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution: False. Take any $\vec{u}$ orthogonal to $\vec{v}$ and $\vec{w} = \vec{v}$. Then $\vec{u}$ is also orthogonal to $\vec{w}$, but $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

For example, let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\vec{u}.\vec{v} = \vec{u}.\vec{w} = 0$, but $0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.