Math 115 Spring 2015: Assignment 3

Solutions

1. Consider the following system of linear equations

$$\begin{array}{rcrcrcrcrcr} & & & & 2x_3 & + & 1x_4 & - & 3x_5 & = & 2 \\ x_1 & + & 3x_2 & - & x_3 & & & + & 4x_5 & = & 2 \\ 2x_1 & + & x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & -1. \end{array}$$

   (a) [2 marks] Write the augmented matrix for this system of linear equations.
   Solution:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 2 & 1 & -3 & 2 \\ 1 & 3 & -1 & 0 & 4 & 2 \\ 2 & 1 & -1 & -1 & 2 & -1 \end{array}\right]$$

   (b) [8 marks] Solve this system using elementary row operations (indicate which operations you are using). Write down the set of all solutions to this system.
   Solution:

$$\left[\begin{array}{ccccc|c} 1 & 3 & -1 & 0 & 4 & 2 \\ 0 & -5 & 1 & -1 & -6 & -5 \\ 0 & 0 & 2 & 1 & -3 & 2 \end{array}\right] \begin{array}{l} R1' \leftarrow R2 \\ R2' \leftarrow R3 - 2.R2 \\ R3' \leftarrow R1 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{5} & -\frac{1}{10} & -\frac{11}{10} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} R1'' \leftarrow R1' + \frac{3}{5}R2' + \frac{1}{2}R3' \\ R2'' \leftarrow -\frac{1}{5}R2' \\ R3'' \leftarrow \frac{1}{2}R3' \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{9}{10} & \frac{6}{5} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} R1''' \leftarrow R1'' - \frac{3}{5}R3'' \\ R2''' \leftarrow R2'' + \frac{1}{5}R3'' \\ R3''' \leftarrow R3'' \end{array}$$

   The general solution is

$$\begin{cases} x_1 = -\frac{3}{5} + \frac{2}{5}s + \frac{1}{5}t \\ x_2 = \frac{6}{5} - \frac{3}{10}s - \frac{9}{10}t \\ x_3 = 1 - \frac{1}{2}s + \frac{3}{2}t \\ x_4 = s \\ x_5 = t \end{cases}, \text{ for all } s, t \in \mathbb{R}.$$

2. Consider the system of linear equations with the following augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & q^2 & -p & 1 \\ 0 & q & 0 & p \\ 0 & 0 & p & pq \end{array}\right]$$

   (a) [3 marks] Determine the values that p and q must take for this system to be consistent with exactly one solution.
   Solution: To have a solution, we need to have no row of the form $[0 \cdots 0|b]$ where $b \neq 0$. The first row cannot be such a row (because of the leading 3). The second would need $q = 0$ and $p \neq 0$ to be

inconsistent. The third is $[0 \cdots 0|0]$ if $p = 0$, so it is always consistent. Thus, the system is consistent if $q \neq 0$ or $p = 0$.

If it is consistent, then the solution is unique if it has three pivots. This happens if $q \neq 0$ and $p \neq 0$.

In summary, we need $p \neq 0$ and $q \neq 0$.

(b) [3 marks] If it has exactly one solution (x_1, x_2, x_3), then give x_1, x_2 and x_3 in function of p and q.

Solution:
$x_3 = pq/p = q$,
$x_2 = p/q$,
$x_1 = 1/3.(1 - q^2 x_2 + p x_3) = 1/3.(1 - q^2 p/q + pq) = 1/3$
The solution is thus $(q,\ p/q,\ 1/3)$.

(c) [2 marks] Determine the values that p and q must take for this system to be inconsistent.

Solution: As noted in (a), only the second row can take the form $[0 \cdots 0|b]$ where $b \neq 0$. For that to happen, we need $q = 0$ and $p \neq 0$.

(d) [2 marks] Determine the values that p and q must take for this system to be consistent with infinitely many solutions.

Solution: As in (a), for the system to be consistent we need either $q \neq 0$ or $p = 0$. To have infinitely many solutions, we need a row of the form $[0 \cdots 0|0]$. This occurs in the second row if $q = p = 0$ or in the third if $p = 0$. So it happens whenever $p = 0$ (and then, the system is indeed consistent).