Math 115 Spring 2015: Assignment 5

Solutions

1. (a) [2 marks] Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\vec{y} = A\vec{x}$, where $\vec{y}$ is $\vec{x}$ rotated by an angle of $\frac{2}{3}\pi$ (counterclockwise around the origin), for any $\vec{x} \in \mathbb{R}^2$.

   (b) [2 marks] Find a matrix $B \in \mathbb{R}^{2\times 2}$ such that $\vec{z} = B\vec{y}$, where z_1 is y_1 scaled by a factor 3 and z_2 is y_2 scaled by a factor 2, for any $\vec{y} \in \mathbb{R}^2$.

   (c) [2 marks] Find a matrix $C \in \mathbb{R}^{2\times 2}$ such that $\vec{w} = C\vec{z}$, where $\vec{w}$ is $\vec{z}$ rotated by an angle of $\frac{-2}{3}\pi$ (counterclockwise around the origin, i.e. $\frac{2}{3}\pi$ clockwise), for any $\vec{z} \in \mathbb{R}^2$.

   (d) [2 marks] Find a matrix $G \in \mathbb{R}^{2\times 2}$ such that $\vec{w} = G\vec{x}$, where $\vec{w}$ is $\vec{x}$ that is rotated by $\frac{2}{3}\pi$ and then scaled with factors 3 and 2, and then rotated by $\frac{-2}{3}\pi$. Note that this amounts to performing on $\vec{x}$ all three transformations found in points (a), (b) and (c), successively.

Solution:

$$
\begin{aligned}
A &= \begin{bmatrix} \cos\left(\frac{2}{3}\pi\right) & -\sin\left(\frac{2}{3}\pi\right) \\ \sin\left(\frac{2}{3}\pi\right) & \cos\left(\frac{2}{3}\pi\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \\
B &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \\
C &= \begin{bmatrix} \cos\left(-\frac{2}{3}\pi\right) & -\sin\left(-\frac{2}{3}\pi\right) \\ \sin\left(-\frac{2}{3}\pi\right) & \cos\left(-\frac{2}{3}\pi\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \\
G &= C \cdot B \cdot A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{2} & -3\frac{\sqrt{3}}{2} \\ \sqrt{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{9}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{11}{4} \end{bmatrix}
\end{aligned}
$$

2. [3 marks] Find a matrix $A \in \mathbb{R}^{2\times 2}$ that has no zero elements, such that $\vec{x} = A^k\vec{x}$. Note that $A^k = A \cdot A \cdots A$ where there are k factors A. For example $A^3 = A \cdot A \cdot A$. **Hint:** Think about a geometric transformation that, when applied k times on the vector $\vec{x}$, gives back the vector $\vec{x}$ itself. k may appear in some form in the matrix.

Solution: Applying k rotations of $\frac{2\pi}{k}$ radians on a vector yields the same vector. So A is given by

$$A = \begin{bmatrix} \cos\left(\frac{2\pi}{k}\right) & -\sin\left(\frac{2\pi}{k}\right) \\ \sin\left(\frac{2\pi}{k}\right) & \cos\left(\frac{2\pi}{k}\right) \end{bmatrix}.$$

This matrix has no zero elements for $k \geq 3$.

Bonus points: For $k = 2$,

$$A = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has zero elements. But in this case, we only need $A^2 = AA = I$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In other words, we have the four equations

$$\begin{cases} a^2+bc &= 1 \\ ab+bd &= 0 \\ ca+dc &= 0 \\ cb+d^2 &= 1 \end{cases}$$

The last equation gives $bc = 1 - d^2$. Using that in the first equation gives $a^2 + 1 - d^2 = 1$ so $a^2 = d^2$, i.e. $|a| = |d|$. The second equation is $b(a+d) = 0$ and the third is $c(a+d) = 0$. We want $a, b, c, d \neq 0$ so we need $a = -d$ and $bc = 1 - a^2$. For example, we can choose $a = 2$, $b = -1$, $c = 3$ and $d = -2$:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$

3. The matrix $A \in \mathbb{R}^{4\times 7}$ and its reduced row echelon form B are given as follows:

$$A = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 1 & -5 \\ 0 & 0 & -1 & 5 & -9 & -1 & 4 \\ 3 & -9 & -1 & 8 & 3 & 1 & -5 \\ -1 & 3 & 1 & -6 & 5 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & -5 & 9 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [2 marks] Determine a basis for the columnspace of A.

Solution: One basis of col(A) is given by the columns of A that correspond to columns of B having a leading one. Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis of col(A).

(b) [2 marks] Determine a basis for the rowspace of A.

Solution: One basis of row(A) is given by the nonzero rows of B. Therefore,

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 9 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -6 \end{bmatrix} \right\}$$

is a basis of row(A).

(c) [5 marks] The set $S = \{x \in \mathbb{R}^7 | A\vec{x} = \vec{0}\}$ is the set of all solutions to the system $A\vec{x} = \vec{0}$. This set S is a subspace. Determine a basis for S. **Hint:** Find the general solution to $A\vec{x} = \vec{0}$, and write it as a vector equation.

Solution: We write the RREF of the system $A\vec{x} = \vec{0}$. Because the right-hand side is zero in the system, it will be zero in its RREF too. The other coefficients in the RREF will simply be the coefficients in the RREF of A:

$$\left[\begin{array}{ccccccc|c} 1 & -3 & 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 9 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution to this system is

$$\left\{ \begin{array}{ccrrrr} x_1 & = & 3x_2 & -1x_4 & -4x_5 & -1x_7 \\ x_3 & = & & 5x_4 & -9x_5 & -2x_7 \\ x_6 & = & & & & 6x_7 \end{array} \right. ,$$

where x_2, x_4, x_5, x_7 are free variables. As a vector equation, we can write this subspace as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} q + \begin{bmatrix} -1 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ -9 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 6 \\ 1 \end{bmatrix} t, \quad \text{for all } q, r, s, t \in \mathbb{R}.$$

It is easy to verify that the only solution for $\vec{x} = 0$ is $q = r = s = t = 0$, so the four vectors above are linearly independent. Therefore,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -9 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 6 \\ 1 \end{bmatrix} \right\}$$

is a basis of S.