Math 115 Spring 2015: Assignment 6

Solutions

1. [5 marks] Let $G = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix}$. Compute G^{-1}.

   Solution: We compute the RREF of $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$. and obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & -5 & 0 & 2 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$. so $G^{-1} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$.

2. [4 marks] Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 \\ 2 & 3 & 5 & 1 & 2 \\ 4 & 1 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

   Compute $\det(A)$. **Hint:** choose carefully the columns or rows to expand in order to reduce your work.

   Solution: We notice that the fourth column of A has only one element different from zero. It thus seems like an good choice for expansion: $\det(A) = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44} + a_{54}C_{54} = 1.C_{34}$. The cofactor C_{34} evaluates to $(-1)^{3+4}\det(A(3,4))$ where $A(3,4)$ is the matrix obtained from A by removing the third row and the fourth column. Thus,

$$\det(A) = C_{34} = -1.\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix}\right).$$

   The first row of $A(3,4)$ only has one nonzero, so we compute the determinant of $A(3,4)$ using that row. We denote by C'_{ij} the cofactors of $A(3,4)$, and get

$$\det(A) = -1.(1.C'_{11} + 0.C'_{12} + 0.C'_{13} + 0.C'_{14}) = -1.C'_{11} = -1.(-1)^{1+1}.\det\left(\begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right).$$

   Again, we only have one cofactor to compute. This time, we use the third column for determinant expansion.

$$\det(A) = -1.1.(-1)^{1+3}.\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right).$$

Since the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $(ad - bc)$, we get

$$\det(A) = -1.(1.1 - 0.1) = -1$$

3. [5 marks] Let

$$B = \begin{bmatrix} 1 & 4 & 5 & 3 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Notice that B is upper-triangular (i.e. all elements below the diagonal are zero). Use the cofactor expansion of determinants to show (on this example) that the $\det(B)$ is simply the product of the diagonal elements of B.

Solution: If we always use the first column for expansion, we get

$$\det(B) = b_{11}C_{11} + b_{21}C_{21} + b_{31}C_{31} + b_{41}C_{41} = b_{11}C_{11} = b_{11}(-1)^{1+1}\det(B(1,1)).$$

Let $B' = B(1,1)$. Notice that B' is upper-triangular as well, so

$$\det(B') = \det\left(\begin{bmatrix} 2 & 3 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 4 \end{bmatrix}\right) = b'_{11}C'_{11} + b'_{21}C'_{21} + b'_{31}C'_{31} = b'_{11}C'_{11} = b_{22}(-1)^{1+1}\det(B'(1,1)).$$

Finally, let $B'' = B'(1,1)$. Again, B'' is upper-triangular, so

$$\det(B'') = \det\left(\begin{bmatrix} 3 & 7 \\ 0 & 4 \end{bmatrix}\right) = b''_{11}b''_{22} - 0.b''_{12} = b_{33}b_{44}.$$

Collecting all our result, we see that

$$\det(B) = b_{11}b_{22}b_{33}b_{44} = 1.2.3.4 = 24.$$

4. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

   (a) [3 marks] If A and B are $n \times n$ invertible matrices, then $A + B$ is also invertible.

   Solution: Take, for example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   The determinants of those matrices are both not zero ($\det(A) = 1$, $\det(B) = -1$), so they are invertible.

But

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not invertible because $\det(A + B) = 0$.

(b) [3 marks] If A and B are $n \times n$ invertible matrices and $(AB)^2 = A^2B^2$, then $AB = BA$.

Solution: First, we note that $(AB)^2 = (AB)(AB) = ABAB$, and $A^2B^2 = AABB$, so we know that

$$ABAB = AABB.$$

Since A and B are invertible, they have inverses A^{-1} and B^{-1}. If we premultiply both side of the above equation by A^{-1}, we obtain

$$A^{-1}ABAB = A^{-1}AABB,$$

i.e.

$$BAB = ABB.$$

Now, we postmultiply both sides by B^{-1} and get

$$BABB^{-1} = ABBB^{-1},$$

i.e.

$$BA = AB.$$