Math 115 Spring 2015: Assignment 7

Solutions

1. [5 marks] Let

$$A = \begin{bmatrix} 4 & 2a+2b & 2a \\ 2 & a+b+1 & a+b \\ 2 & a+b+1 & a+b+1 \end{bmatrix},$$

where $a, b \in \mathbb{R}$. The determinant of A is a constant independent of a and b. Find its value. **Hint:** Compute the REF of A.

Solution: We put A in REF and obtain (for example)

$$\begin{bmatrix} 4 & 2a+2b & 2a \\ 2 & a+b+1 & a+b \\ 2 & a+b+1 & a+b+1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2a+2b & 2a \\ 0 & 1 & b \\ 0 & 1 & b+1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2a+2b & 2a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

We only performed row operations of the type "adding a multiple of a row to another" so we did not affect the determinant. The last matrix is triangular so its determinant is the product of the diagonal elements. Therefore, $\det(A) = 4.1.1 = 4$.

2. [5 marks] Let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ t \cdot b_{31} & t \cdot b_{32} & t \cdot b_{33} \end{bmatrix}$$

for $t \in \mathbb{R}$. Use Cramer's rule to prove that

$$\text{if} \quad B^{-1} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \quad \text{then} \quad C^{-1} = \begin{bmatrix} g_{11} & g_{12} & \frac{1}{t} \cdot g_{13} \\ g_{21} & g_{22} & \frac{1}{t} \cdot g_{23} \\ g_{31} & g_{32} & \frac{1}{t} \cdot g_{33} \end{bmatrix}.$$

Solution: By Cramer's rule,

$$B^{-1} = \frac{1}{det(B)} \operatorname{cof}(B)^T \quad \text{and}$$
$$C^{-1} = \frac{1}{det(C)} \operatorname{cof}(C)^T$$

First, note that C is obtained by multiplying the third row of B by t. Therefore, $\det(C) = t \det(B)$.

Furthermore, the third row of $\operatorname{cof}(C)$ is the same as the third row of $\operatorname{cof}(B)$, because the first and second row of B and C are the same. For the other two rows of $\operatorname{cof}(C)$, the values are the corresponding values of $\operatorname{cof}(B)$ multiplied by t. Indeed, they correspond to the determinant of the same submatrix, except that one row has been multiplied by t in the case of $\operatorname{cof}(C)$.

Therefore, we obtain that

$$C^{-1} = \frac{1}{t.\det(B)} \begin{bmatrix} tC_{11} & tC_{21} & C_{31} \\ tC_{11} & tC_{21} & C_{31} \\ tC_{11} & tC_{21} & C_{31} \end{bmatrix} = \frac{1}{\det(B)} \begin{bmatrix} C_{11} & C_{21} & \frac{1}{t}C_{31} \\ C_{11} & C_{21} & \frac{1}{t}C_{31} \\ C_{11} & C_{21} & \frac{1}{t}C_{31} \end{bmatrix}$$

where C_{ij} are the cofactors of B.

3. [5 marks] For each of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, determine whether or not it is an eigenvector of E. If so, determine its corresponding eigenvalue.

$$E = \begin{bmatrix} 12 & -18 & 6 \\ 13 & -17 & 6 \\ 11 & -9 & 4 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}.$$

Solution:

$$E \cdot \vec{v}_1 = \begin{bmatrix} 12 & -18 & 6 \\ 13 & -17 & 6 \\ 11 & -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ 8 \end{bmatrix} = -2 \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}.$$

So $\vec{v}_1$ is an eigenvector of E (the associated eigenvalue is -2).

$$E \cdot \vec{v}_2 = \begin{bmatrix} 12 & -18 & 6 \\ 13 & -17 & 6 \\ 11 & -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 36 \\ 35 \\ 21 \end{bmatrix}$$

which is not a scalar multiple of $\vec{v}_2$. So $\vec{v}_2$ is not an eigenvector of E.

$$E \cdot \vec{v}_3 = \begin{bmatrix} 12 & -18 & 6 \\ 13 & -17 & 6 \\ 11 & -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is a scalar multiple of $\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, with scalar multiplier zero. So $\vec{v}_3$ is an eigenvector of E (with associated eigenvalue 0).

4. [5 marks] Prove that if λ is an eigenvalue of $D \in \mathbb{R}^{n \times n}$, then λ^2 is an eigenvalue of $D \cdot D$.
Solution: Since λ is an eigenvalue of $D \in \mathbb{R}^{n \times n}$, there exist a corresponding eigenvector $\vec{v} \in \mathbb{R}^n$, i.e. a vector such that $D\vec{v} = \lambda\vec{v}$.

Let us compute $D \cdot D \cdot \vec{v}$:

$$\begin{aligned}
(D \cdot D) \cdot \vec{v} &= D \cdot (D \cdot \vec{v}) \\
&= D \cdot (\lambda \vec{v}) \\
&= \lambda (D \cdot \vec{v}) \\
&= \lambda (\lambda \vec{v}) \\
&= \lambda^2 \vec{v}.
\end{aligned}$$

Therefore, λ^2 is an eigenvalue of $(D \cdot D)$ (and $\vec{v}$ is a corresponding eigenvector).