Math 115 Spring 2015: Assignment 9

Solutions

1. [5 marks] Let

$$\vec{v}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Knowing that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, write the vector $\vec{x} = \begin{bmatrix} 6 \\ -6 \\ 15 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Solution: Note that since $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$, it is clear that $\vec{x} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Therefore, because $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthonormal, we can write

$$\vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3$$

where

$$t_1 = \vec{x} \cdot \vec{v}_1, \quad t_2 = \vec{x} \cdot \vec{v}_2, \quad t_3 = \vec{x} \cdot \vec{v}_3.$$

We have

$$\begin{aligned} t_1 &= \begin{bmatrix} 6 \\ -6 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = 4 - 4 + 5 = 5 \\ t_2 &= \begin{bmatrix} 6 \\ -6 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = 4 + 2 - 10 = -4 \\ t_3 &= \begin{bmatrix} 6 \\ -6 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = 2 + 4 + 10 = 16 \end{aligned}$$

so

$$\vec{x} = 5\vec{v}_1 - 4\vec{v}_2 + 16\vec{v}_3.$$

2. [5 marks] Let $S = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$, with

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find a basis of $S^\perp$. **Hint:** A vector is orthogonal to S if and only if it is orthogonal to both $\vec{u}_1$ and $\vec{u}_2$.

Solution: For a vector $\vec{x} \in \mathbb{R}^4$ to be orthogonal to S, it has to satisfy the following equations:

$$\begin{cases} \vec{x} \cdot \vec{u}_1 = 0 \\ \vec{x} \cdot \vec{u}_2 = 0 \end{cases}$$

This can be written as

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vec{x} = \vec{0}.$$

We solve this system by writing its augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and obtain that

$$\vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis of $S^{\perp}$.

3. [10 marks] Let

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Find an orthonormal basis of $\text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

Solution: For the first vector, we set

$$\vec{v}_1 = \frac{\vec{w}_1}{|\vec{w}_1|} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2 + 0^2}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}.$$

For the second vector, we have

$$\begin{aligned}
\vec{u}_2 &= \vec{w}_2 - (\vec{w}_2 \cdot \vec{v}_1)\vec{v}_1 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}
\end{aligned}$$

and

$$\vec{v}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}} \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} = \frac{3}{\sqrt{15}} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix}.$$

For the third vector, we have

$$\begin{aligned}
\vec{u}_3 &= \vec{w}_3 - (\vec{w}_3 \cdot \vec{v}_1)\vec{v}_1 - (\vec{w}_3 \cdot \vec{v}_2)\vec{v}_2 \\
&= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \sqrt{3} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} - \frac{\sqrt{3}}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix}
\end{aligned}$$

and

$$\vec{v}_3 = \frac{\vec{u}_3}{|\vec{u}_3|} = \frac{1}{\sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(-\frac{3}{5}\right)^2}} \begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix} = \frac{3}{\sqrt{15}} \begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix} = \frac{5}{\sqrt{35}} \begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{35}} \\ \frac{4}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \\ -\frac{3}{\sqrt{35}} \end{bmatrix}.$$

An orthonormal basis of $\operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is thus given by

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ \frac{2}{\sqrt{15}} \\ \frac{3}{\sqrt{15}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{35}} \\ \frac{4}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \\ -\frac{3}{\sqrt{35}} \end{bmatrix} \right\}.$$

4. [10 marks] Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix},$$

i.e. find an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Solution: First, we find the eigenvalues of A. The characteristic polynomial of A is

$$\begin{aligned}
& \det(A-\lambda I) \\
= \;& \det\begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{bmatrix} \\
= \;& (1-\lambda)\det\begin{bmatrix} 4-\lambda & -4 \\ -4 & 4-\lambda \end{bmatrix} + 2\det\begin{bmatrix} -2 & -4 \\ 2 & 4-\lambda \end{bmatrix} + 2\det\begin{bmatrix} -2 & 4-\lambda \\ 2 & -4 \end{bmatrix} \\
= \;& (1-\lambda)(16+\lambda^2-8\lambda-16) + 2(-8+2\lambda+8) + 2(8-8+2\lambda) \\
= \;& -\lambda^3 + 9\lambda^2 = -\lambda^2(\lambda-9)
\end{aligned}$$

First eigenvalue: $\lambda = 0$.
The corresponding eigenvectors $\vec{u}$ satisfy $A\vec{u} = 0\vec{u}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ -2 & 4 & -4 & 0 \\ 2 & -4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We thus have $u_1 = 2u_2 - 2u_3$ so

$$\vec{u} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

We need an orthonormal basis of this eigenspace. So we apply Gram-Schmidt on

$$\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{w}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

We get

$$\vec{r}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

and

$$\vec{q}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}\right)\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{\sqrt{5}}\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{8}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$$

which we normalize:

$$\vec{r}_2 = \begin{bmatrix} -\frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}$$

Second eigenvalue: $\lambda = 9$.

The corresponding eigenvectors $\vec{v}$ satisfy $A\vec{v} = 9\vec{v}$. The augmented matrix of the system $(A - 9I)\vec{v} = \vec{0}$ is

$$\left[\begin{array}{ccc|c} -8 & -2 & 2 & 0 \\ -2 & -5 & -4 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We thus have $v_1 = \frac{1}{2}v_3$ and $v_2 = -v_3$ so

$$\vec{v} = s \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

In order to obtain an orthonormal basis of this eigenspace, we normalize the basis vector and get

$$\vec{r}_3 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Constructing P: We have

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} & -\frac{2}{3} \\ 0 & \frac{5}{\sqrt{45}} & \frac{2}{3} \end{bmatrix}$$

And we could verify that

$$P^T AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

5. (a) [3 marks] Show that the following statement is false by providing a counter-example:

   Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$ and $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for some positive integer $k < n$. Then $S^\perp = \text{span}\{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$.

   (b) [2 marks] What additional property would $\{\vec{v}_1, \ldots, \vec{v}_n\}$ need for the statement to be true? (no proof necessary)

Solution: (a) Take

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The vectors are linearly independent and $\text{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$, so $\{\vec{v}_1, \vec{v}_2\}$ is a basis of $\mathbb{R}^2$. However,

$$\text{span}\{\vec{v}_1\}^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \neq \text{span}\{\vec{v}_2\}.$$

(b) If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ was an *orthonormal* basis of $\mathbb{R}^n$, then the statement would be true.

6. [5 marks] Prove that the determinant of an orthogonal matrix is always 1 or -1. **Hint:** Recall that for any square matrix A, we have $\det(A) = \det(A^T)$. Also, if A is invertible, then $A^{-1}A = I$.

Solution: Let A be an orthogonal matrix, we have $A^{-1} = A^T$ so $A^T A = I$. Taking the determinant of both sides yields

$$\begin{aligned} A^T A &= I \\ \det(A^T A) &= \det(I) \\ \det(A^T)\det(A) &= 1 \\ \det(A)\det(A) &= 1 \\ (\det(A))^2 &= 1 \end{aligned}$$

Therefore $\det(A) = \pm\sqrt{1} = \pm 1$.