# Stacks and queues

# Abstract data types
# and
# data structures

- An abstract data type
  - Specifies supported operations

- A data structure is an implementation of an abstract data type
  - Specifies data layout in memory
  - Specifies algorithms for operations

# Lists

- support storing multiple elements together

  and optionally append, insert, delete, random access, …
- implementations:
  - dynamic arrays

    everything $O(1)$ in practice except insert/delete $O(n)$
  - linked lists

    everything $O(1)$ (but slower than arrays) except random access $O(n)$

# Stacks / LIFO

- A stack is an ordered collection of elements
- supports two operations:
  - “push”: add an element
  - “pop”: retrieve-and-remove the last-added element

    $\Rightarrow$ last in, first out (LIFO)

# Static array implementation of a stack

- Useful only when there is a hard limit on the number of elements
- We maintain a static array
- and a **stack pointer** (or **top index**)
- this is how “the” stack is implemented

  (for storing function arguments, local variables and return addresses)


↑ stack pointer

→ push A

push B

push C

pop

pop

push E

push F

push G

pop

pop

pop

pop

.


push A

→ push B

push C

pop

pop

push E

push F

push G

pop

pop

pop

pop

.


push A

push B

→ push C

pop

pop

push E

push F

push G

pop

pop

pop

pop

.


push A

push B

push C

→ pop

pop

push E

push F

push G

pop

pop

pop

pop

.


push A

push B

push C

pop → **C**

→ pop

push E

push F

push G

pop

pop

pop

pop

.


↑ stack pointer

```
    push A
    push B
    push C
    pop      → C
    pop      → B
→   push E
    push F
    push G
    pop
    pop
    pop
    pop
    .
```


↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

→ push F

push G

pop

pop

pop

pop

.


↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

→ push G

pop

pop

pop

pop

.


↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

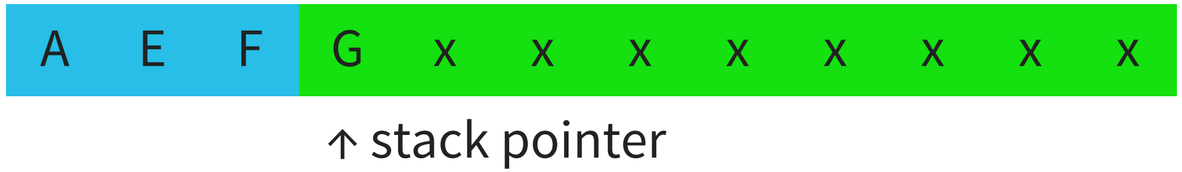
→ pop

pop

pop

pop

.




push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

→ pop

pop

pop

.


push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

pop → **F**

→ pop

pop

.


↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

pop → **F**

pop → **E**

→ pop

.


↑ stack pointer

push A

push B

push C

pop → **C**

pop → **B**

push E

push F

push G

pop → **G**

pop → **F**

pop → **E**

pop → **A**

→ .

# Linked list implementation of a stack

- Pro: No hard limit on number of elements

- Con:
  - Memory allocation for every **push**
  - Memory freed for every **pop**

# Dynamic array implementation of a stack

- Pros:
  - No hard limit on number of elements
  - Memory management overhead is small
- Con:
  - No pointer stability


↑ stack pointer

→ push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

.


push A

→ push B

push C

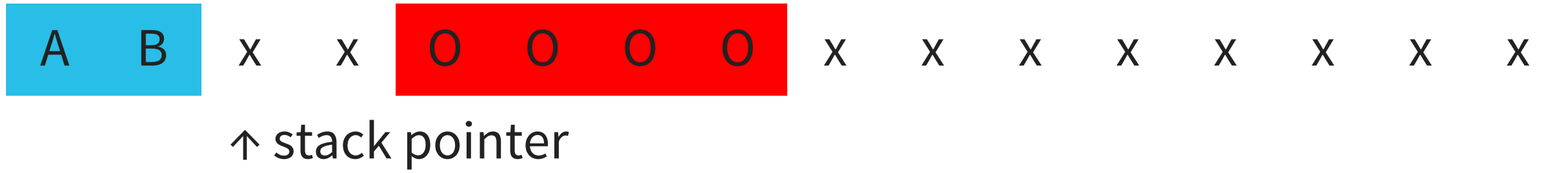
p = address of C

push D

push E

change C into C’ using p

.




push A

push B

→ push C

p = address of C

push D

push E

change C into C’ using p

.


push A

push B

push C

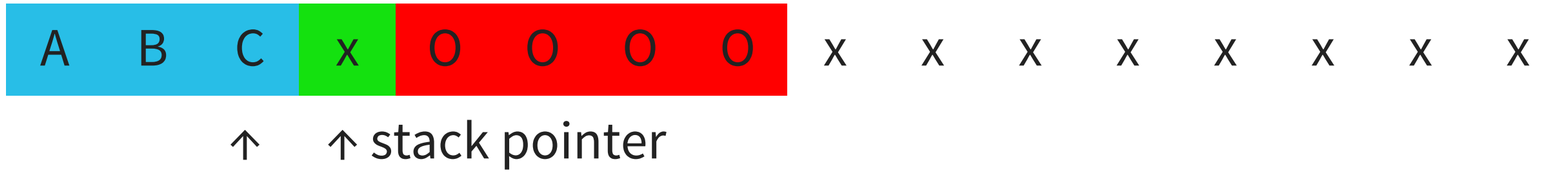
→ p = address of C

push D

push E

change C into C’ using p

.




push A

push B

push C

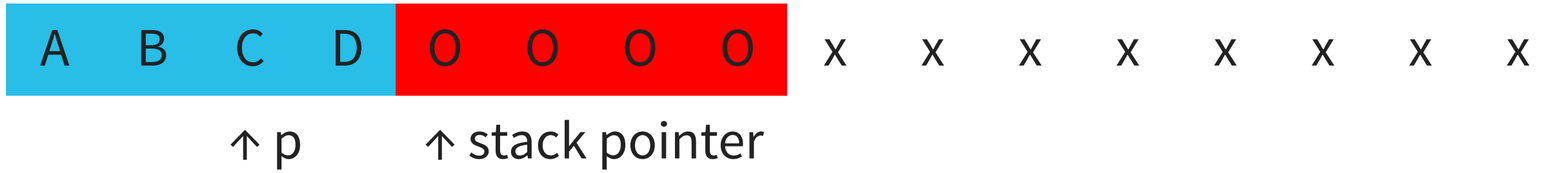
p = address of C

→ push D

push E

change C into C’ using p

.




push A

push B

push C

p = address of C

push D

→ push E

change C into C’ using p

.


push A

push B

push C

p = address of C

push D

push E

→ change C into C’ using p

.


push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

→ .

# Stack implementations

| Implementation | Size | Requires allocations | Pointer stability |
|---|---|---|---|
| static array | constant | no | yes |
| dynamic array | can grow | when growing | no |
| linked list | can grow | every push | yes |

# Arena

- Known as arena allocator, region-based allocator, zone-based allocator, obstack
- Implemented as a list of static array stacks


↑ stack pointer

↑ block 0

→ push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0


↑ stack pointer

↑ block 0

push A

→ push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0


↑ stack pointer

↑ block 0

push A

push B

→ push C

p = address of C

push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0


↑ stack pointer

↑ block 0

push A

push B

push C

→ p = address of C

push D

push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0


↑ ↑ stack pointer

↑ block 0

push A

push B

push C

p = address of C

→ push D

push E

change C into C’ using p

push F

pop

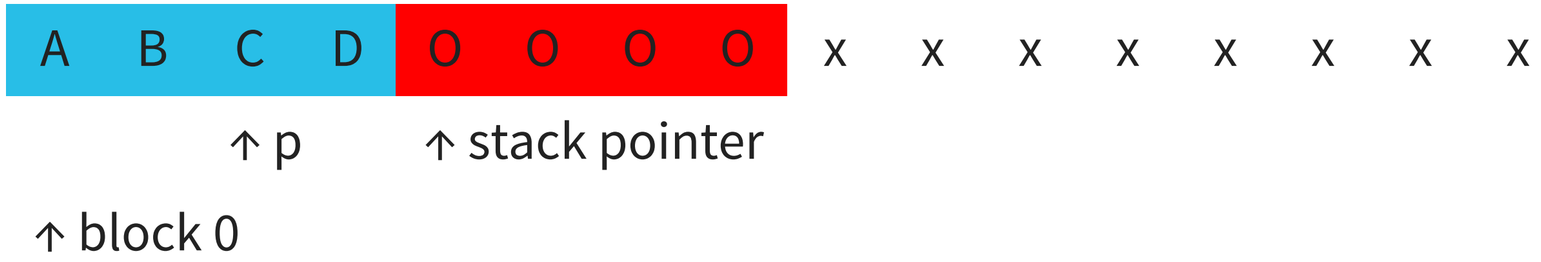
pop

pop

pop

.

list of regions: block 0




push A

push B

push C

p = address of C

push D

→ push E

change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0


push A

push B

push C

p = address of C

push D

push E

→ change C into C’ using p

push F

pop

pop

pop

pop

.

list of regions: block 0

block 1


push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

→ push F

pop

pop

pop

pop

.

list of regions: block 0

block 1


push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

→ pop

pop

pop

pop

.

list of regions: block 0

block 1


push A

push B

push C

p = address of C

push D

push E

change C into C’ using p

push F

pop → F

→ pop

pop

pop

.

list of regions: block 0

block 1


↑ p

↑ stack pointer

↑ block 0

↑ block 1

push A

push B

push C

p = address of C

push D

push E

change C into C' using p

push F

pop → F

pop → E

→ pop

pop

.

list of regions: block 0

block 1


push A

push B

push C

p = address of C

push D

push E

change C into C' using p

push F

pop → F

pop → E

pop → D

→ pop

.

list of regions: block 0


↑ stack pointer

↑ block 0

push A

push B

push C

p = address of C

push D

push E

change C into C' using p

push F

pop → F

pop → E

pop → D

pop → C'

→ .

list of regions: block 0

# Stack implementations

| Implementation | Size | Requires allocations | Pointer stability |
|---|---|---|---|
| static array | constant | no | yes |
| dynamic array | can grow | when growing | no |
| linked list | can grow | every push | yes |
| arena | can grow | when growing | yes |

# More options

Combination of other data structures and indirection can be used, depending on desired properties.

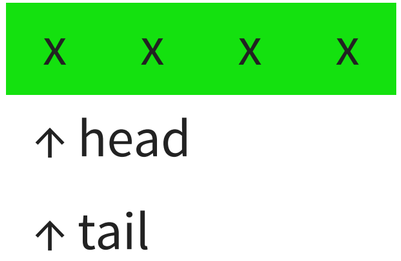
# Queues / FIFO

- A queue is an ordered collection of elements
- supports two operations:
  - “enqueue”: add an element
  - “dequeue”: retrieve-and-remove the earliest-added element

    $\Rightarrow$ first in, first out (FIFO)

# Ring buffer implementation of a queue

- Useful only when there is a hard limit on the number of elements
- We maintain a static array
- and two pointers/indices: **head** and **tail**




→ enqueue A

enqueue B

dequeue

dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

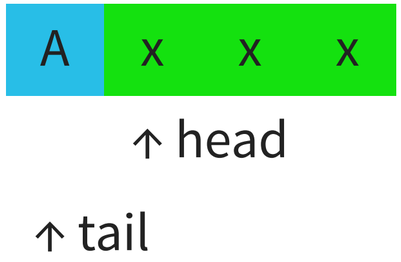
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

→ enqueue B

dequeue

dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

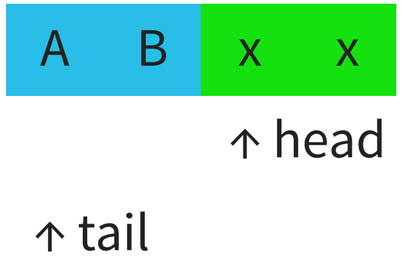
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

→ dequeue

dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

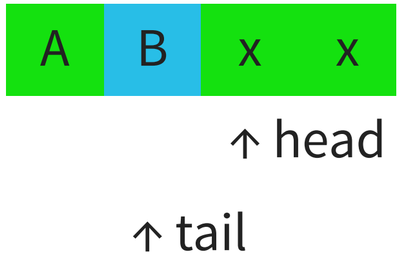
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

→ dequeue

enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

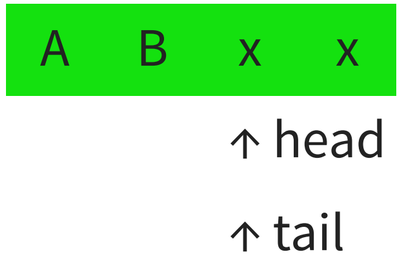
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

→ enqueue C

enqueue D

enqueue E

enqueue F

dequeue

dequeue

dequeue

enqueue G

dequeue

dequeue

.


↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

→ enqueue D

enqueue E

enqueue F

dequeue

dequeue

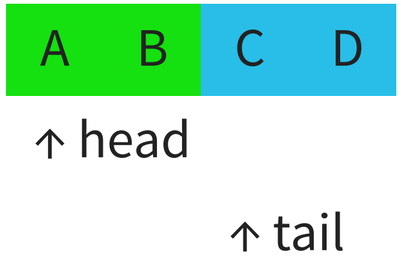
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

→ enqueue E

enqueue F

dequeue

dequeue

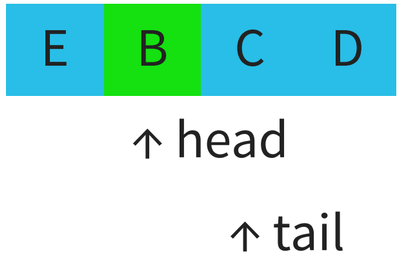
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

→ enqueue F

dequeue

dequeue

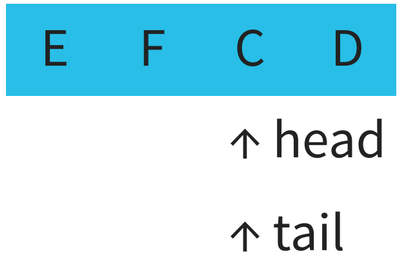
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

→ dequeue

dequeue

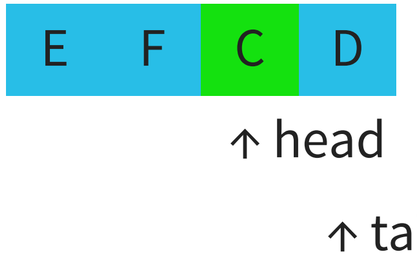
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

→ dequeue

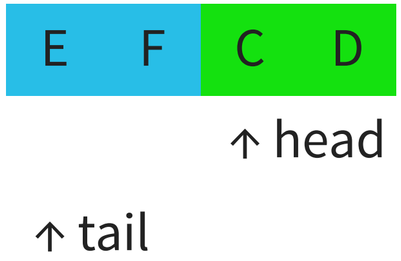
dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

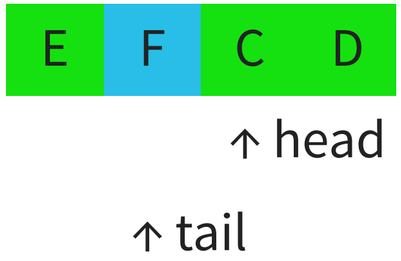
→ dequeue

enqueue G

dequeue

dequeue

.




enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

→ enqueue G

dequeue

dequeue

.


↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

enqueue G

→ dequeue

dequeue

.


↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

enqueue G

dequeue → F

→ dequeue

.


↑ head

↑ tail

enqueue A

enqueue B

dequeue → A

dequeue → B

enqueue C

enqueue D

enqueue E

enqueue F

dequeue → C

dequeue → D

dequeue → E

enqueue G

dequeue → F

dequeue → G

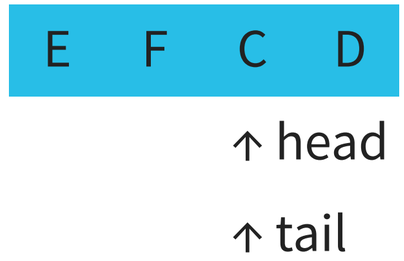
→ .

# Implementation detail

When head == tail:

the queue is empty?


or the queue is full?




- maintain a variable with the number of elements currently in the queue
- or keep incrementing head and tail, and index the static array as

  `array[head % size]` and

  `array[tail % size]`

# Applications of ring buffers

- Audio playback/recording devices
- Video capture devices
- Special case (double-buffering, i.e. size = 2) for computer graphics
- Network devices (routers, switches)

# More options

- use dynamic arrays
- use linked lists
- use indirection
- …

… depending on specific needs

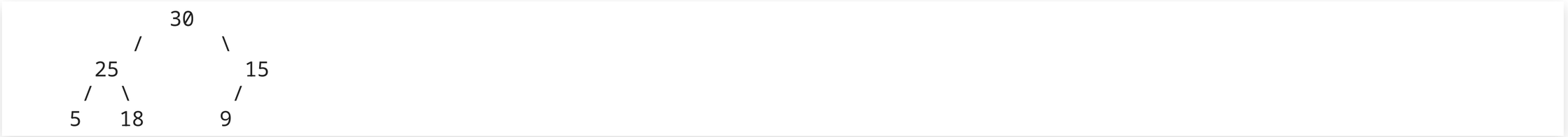
# Priority queues

- A priority queue is a collection of elements, each with an associated priority
- supports two operations:
  - “push”: add an element-priority tuple
  - “pop”: retrieve-and-remove the highest-priority element

# Implementation of a priority queue

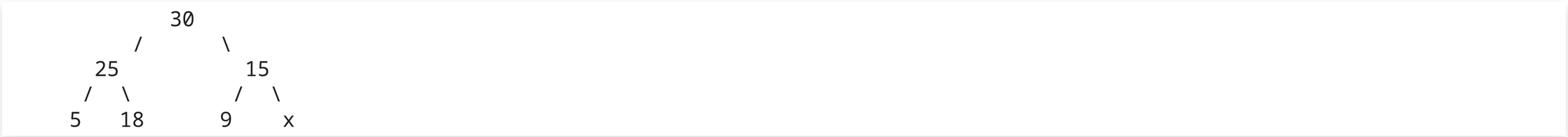
- Store element-priority tuples in an array or in a linked list
- “push”: $O(1)$ of the underlying data structure
- “pop”: scan all elements, find max priority, $O(n)$

# Binary heap implementation of a priority queue



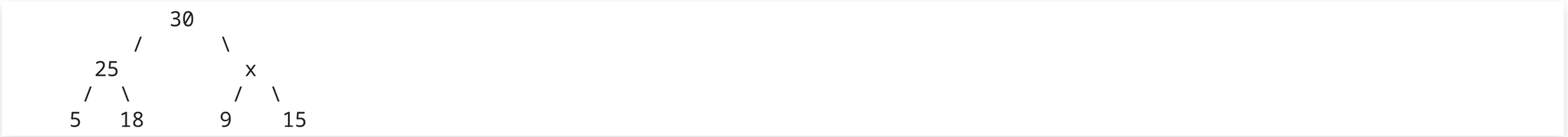

- Binary heaps represent a priority queue as
  - a binary tree (every node has at most two children)
  - that is complete (every level full, except possibly the deepest)
- Every node is labeled by the corresponding element's priority
- Tree has the heap property:
  - Priority of any node $\geq$ priority of its children
  - $\Leftrightarrow$ Priority of any node $\geq$ priority of all its descendants

# Binary heap push




- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
    - the heap property is satisfied, we are done
  - If its priority is higher than its parent's,
    - swap them,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap push




- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
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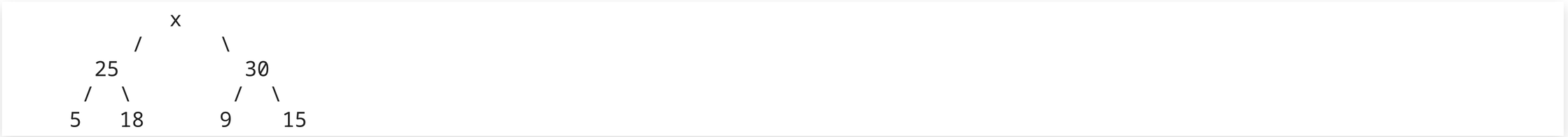
# Binary heap push




- Step 0: Add new element at the first free slot on the deepest level
- Step 1:
  - If its priority is not higher than its parent's,
    - the heap property is satisfied, we are done
  - If its priority is higher than its parent's,
    - swap them,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap push: 14

```
            30
          /    \
        25       15
       /  \     /  \
      5   18   9    14   <---  heap property satisfied
```

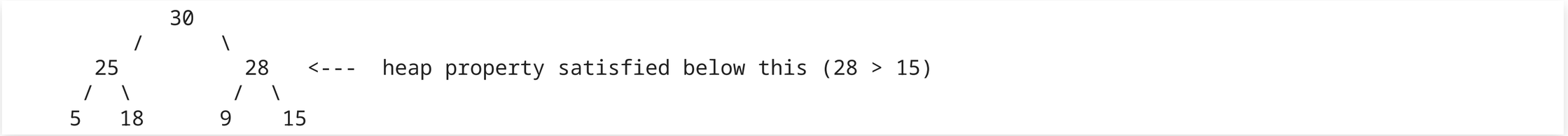
- Add new element at the first free slot on the deepest level
- If it is not higher than its parent's,
  - the heap property is satisfied, we are done

# Binary heap push: 28

```
            30
          /    \
        25       15
       /  \     /  \
      5   18   9    28   <---  heap property NOT satisfied
```

- Add new element at the first free slot on the deepest level
- If its priority is higher than its parent's,
  - swap them

# Binary heap push: 28




- Heap property satisfied below the pushed element's new position
- If its priority is not higher than its new parent's,
  - the heap property is satisfied, we are done

# Binary heap push: 35

```
            30
          /    \
        25       15
       /  \     /  \
      5   18   9    35   <---  heap property NOT satisfied
```

- Add new element at the first free slot on the deepest level
- If its priority is higher than its parent's,
  - swap them

# Binary heap push: 35

```
            30
          /    \
        25       35     <--- heap property NOT satisfied above
       /  \     /  \
      5   18   9    15
```

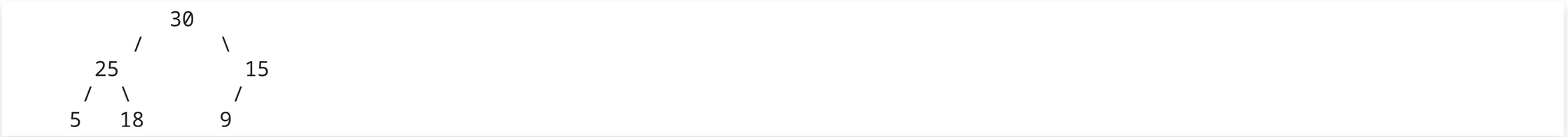
- Heap property satisfied below the pushed element's new position
- If its priority is still higher than its new parent's,
  - swap them

# Binary heap push: 35

```
            35
          /    \
        25       30   <---  heap property satisfied below this (30 > 15)
       /  \     /  \
      5   18   9    15
```

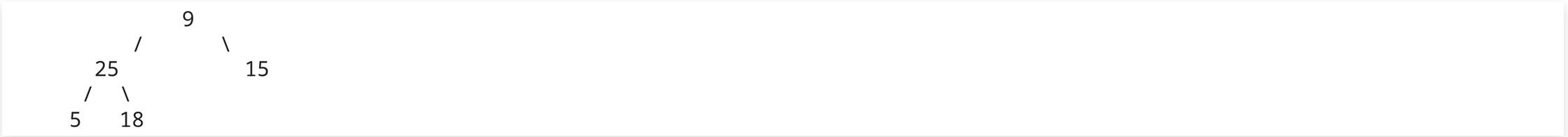
- Heap property satisfied below the pushed element's new position
  - new child was an ancestor of its direct children
- Continue until heap property is satisfied

# Binary heap pop




- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap pop



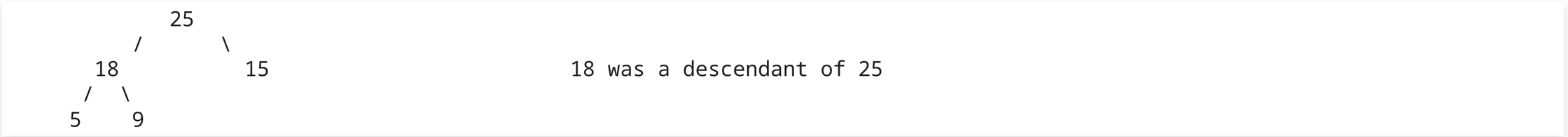

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap pop

```
          25          <--- priority ≥ its children's
        /    \
      9        15       <--- heap property untouched here
    /   \
   5    18
```

- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap pop




- Step 0: Replace root with last element (on deepest level)
- Step 1:
  - If its priority is not lower than its children's,
    - the heap property is satisfied, we are done
  - If its priority is lower than one of its children's,
    - swap with the highest-priority child,
    - go back to Step 1, looking at the pushed element's new position

# Binary heap operations

- Push: $O(\log_2(n))$
- Find max: $O(1)$
- Pop: $O(\log_2(n))$

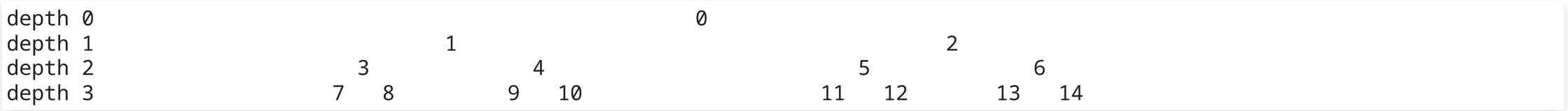
# Array representation of binary heaps

- Binary heaps are complete binary trees
- We can avoid allocation for every “push” by storing nodes in an array
- Depth $\ell$ of the tree has at most $2^\ell$ nodes, $\forall \ell$
- Depth $\ell$ of the tree has exactly $2^\ell$ nodes, except for the deepest level

| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- There are exactly $(2^\ell - 1)$ nodes of with depth $< \ell$

# Storage scheme




| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- If a node has index $j$
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

# Storage scheme

```
depth 0                                   0
depth 1                       1                               2
depth 2           3                   4                   5               6
depth 3       7       8           9       10          11      12      13      14
```

| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- If a node has index $j$
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

# Storage scheme

```
depth 0                                   0
depth 1                       1                                   2
depth 2           3                   4                   5                   6
depth 3       7       8           9       10          11      12          13      14
```

| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- If a node has index $j$
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

# Storage scheme

```
depth 0                                   0
depth 1                       1                               2
depth 2           3                   4                   5               6
depth 3       7       8           9       10          11      12      13      14
```

| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- If a node has index $j$
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

# Storage scheme

```
depth 0                                   0
depth 1                       1                               2
depth 2           3                   4                   5               6
depth 3       7       8           9       10          11      12      13      14
```

| depth | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- If a node has index $j$
- its children are stored at indices $2j + 1$ and $2j + 2$
- its parent is stored at index $\lfloor (j - 1)/2 \rfloor$

# Binary heap with array storage

- Superior to in-memory tree (with pointers)
  - We avoid allocation for every “push”
  - We avoid data dependencies (load node data to get pointer to parent/children)
- Still,
  - Push and pop operations are tough for branch predictor
  - Jumps to indices $(2j+1)$ , $(2j+2)$ or $\lfloor (j-1)/2 \rfloor$

    not cache-friendly for large $j$

# Priority queue: special case

- Assume that
  - priorities are distinct integers $p \in \{0, \ldots, P-1\}$
  - we always push at a priority $\leq$ current max priority
- Then,
  - we allocate a static array of size $P$
  - Push: store in array at index $p$ $O(1)$
  - Pop: sweep array backwards $O(P/n)$ amortized

| priority | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| element | | | | | | | | | | | | | | | | |
| max | | | | | | | | | | | | | | | | ↑ |

→ push A, 12

push B, 5

push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

pop

push H, 1

pop

pop

.


push A, 12

→ push B, 5

push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

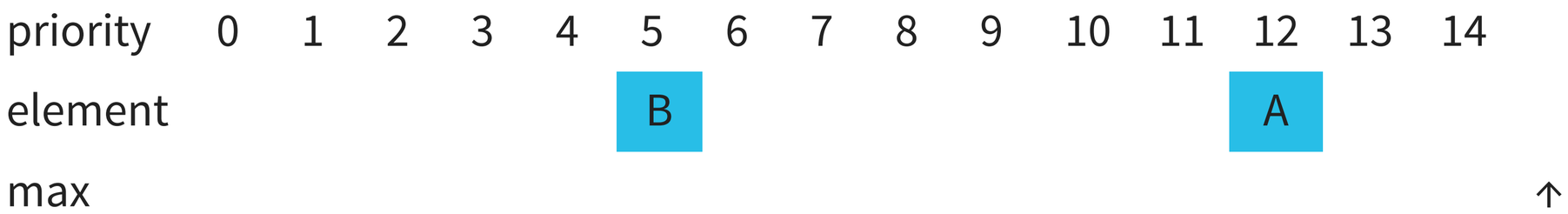
pop

push H, 1

pop

pop

.




push A, 12

push B, 5

→ push C, 9

pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

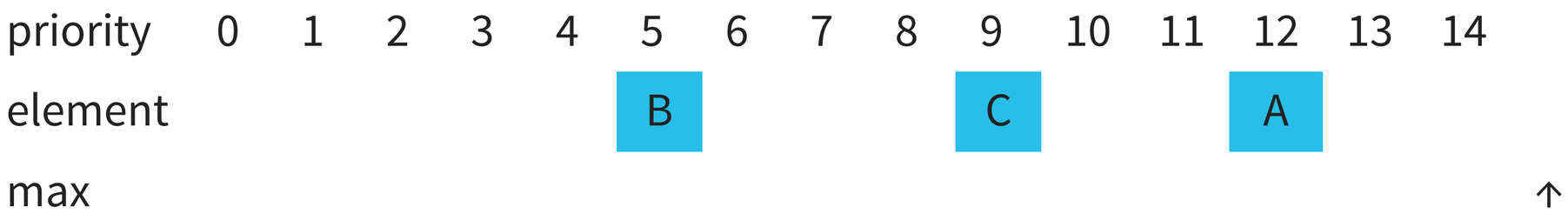
pop

push H, 1

pop

pop

.




push A, 12

push B, 5

push C, 9

→ pop

push E, 3

push F, 10

pop

push G, 4

pop

pop

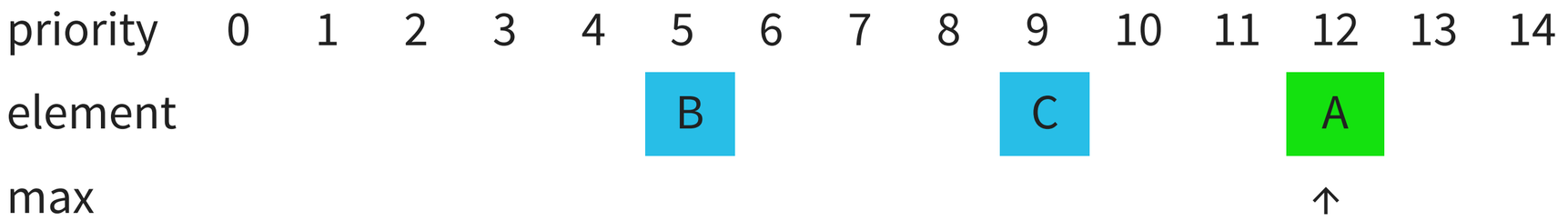
pop

push H, 1

pop

pop

.




push A, 12

push B, 5

push C, 9

pop → A

→ push E, 3

push F, 10

pop

push G, 4

pop

pop

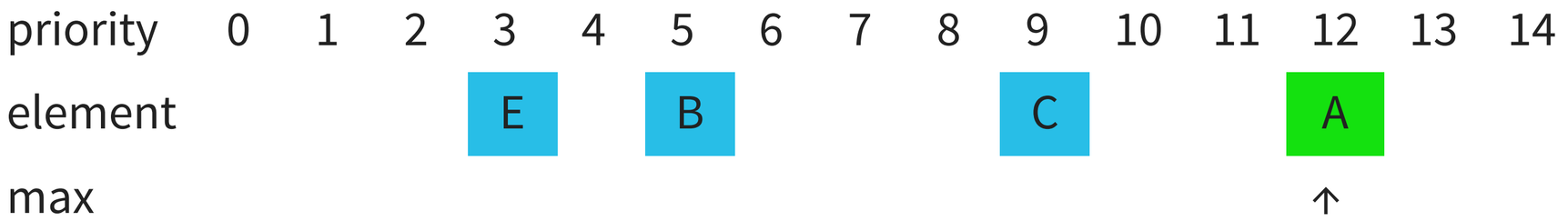
pop

push H, 1

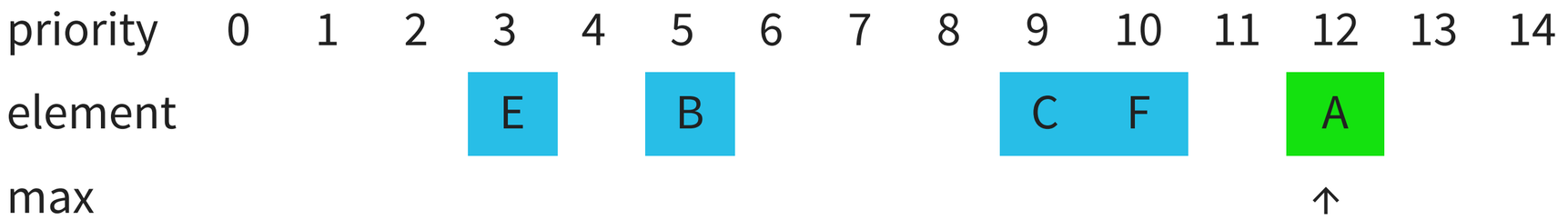
pop

pop

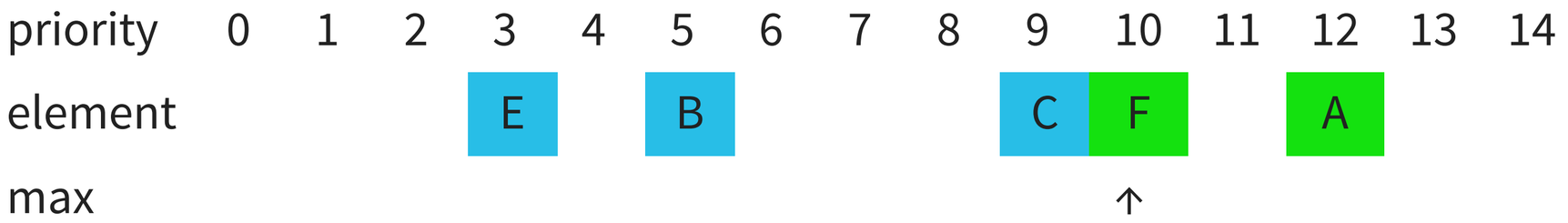
.



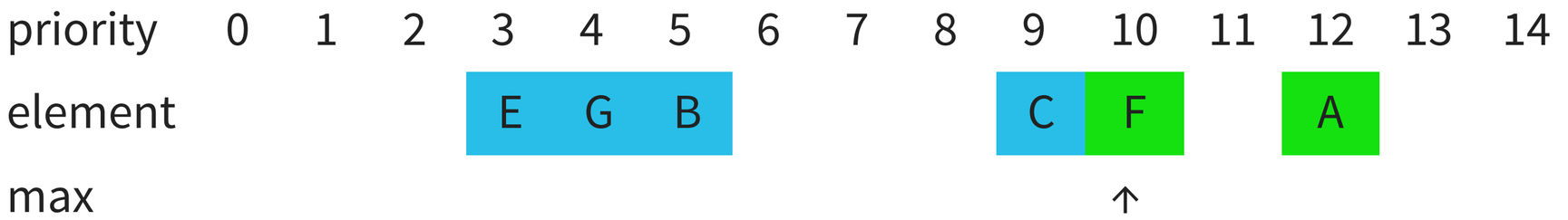

```
    push A, 12
    push B, 5
    push C, 9
    pop          → A
    push E, 3
→   push F, 10
    pop
    push G, 4
    pop
    pop
    pop
    push H, 1
    pop
    pop
    .
```



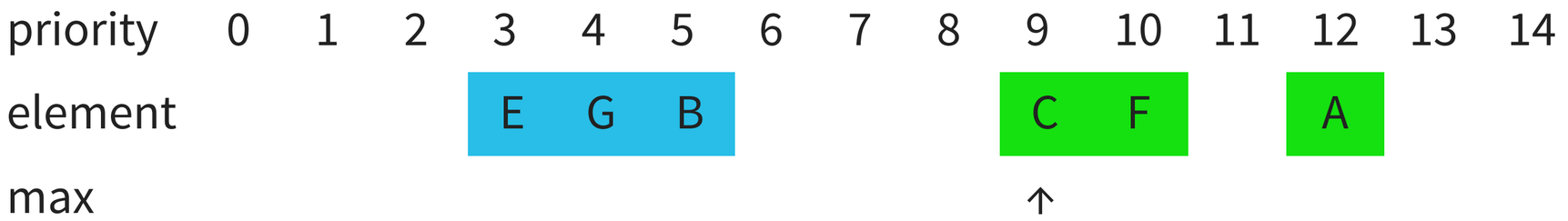

push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
→ pop
push G, 4
pop
pop
pop
push H, 1
pop
pop
.




```
    push A, 12
    push B, 5
    push C, 9
    pop          → A
    push E, 3
    push F, 10
    pop          → F
→   push G, 4
    pop
    pop
    pop
    push H, 1
    pop
    pop
    .
```




push A, 12
push B, 5
push C, 9
pop → A
push E, 3
push F, 10
pop → F
push G, 4
→ pop
pop
pop
push H, 1
pop
pop
.




push A, 12

push B, 5

push C, 9

pop → A

push E, 3

push F, 10

pop → F

push G, 4

pop → C

→ pop

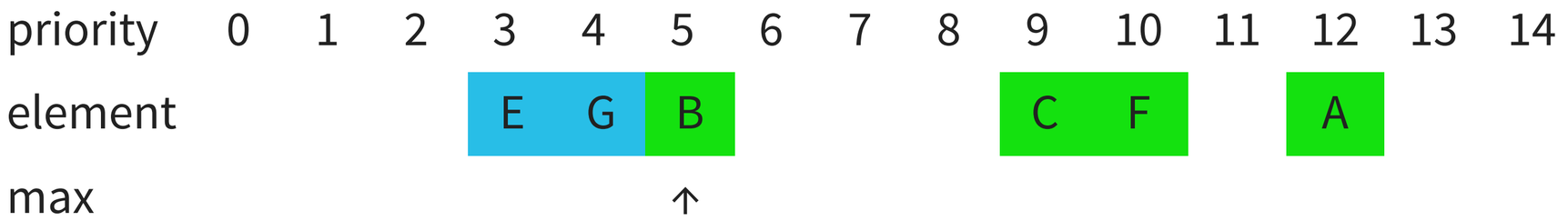
pop

push H, 1

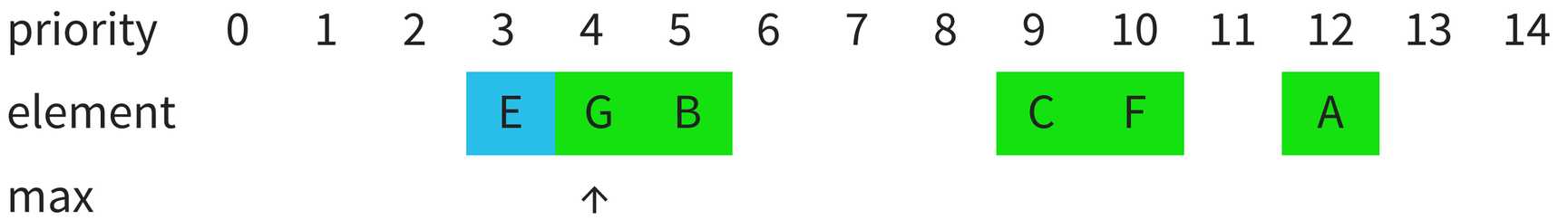
pop

pop

.




```
    push A, 12
    push B, 5
    push C, 9
    pop          → A
    push E, 3
    push F, 10
    pop          → F
    push G, 4
    pop          → C
    pop          → B
→   pop
    push H, 1
    pop
    pop
    .
```




push A, 12

push B, 5

push C, 9

pop → A

push E, 3

push F, 10

pop → F

push G, 4

pop → C

pop → B

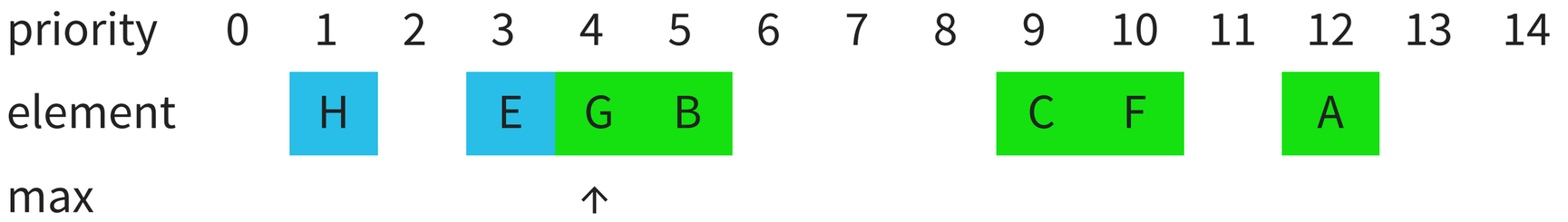
pop → G

→ push H, 1

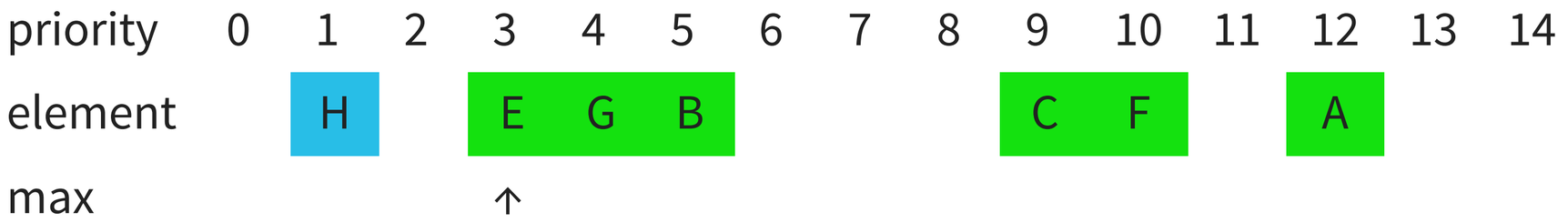
pop

pop

.




```
    push A, 12
    push B, 5
    push C, 9
    pop          → A
    push E, 3
    push F, 10
    pop          → F
    push G, 4
    pop          → C
    pop          → B
    pop          → G
    push H, 1
→   pop
    pop
    .
```




push A, 12

push B, 5

push C, 9

pop → A

push E, 3

push F, 10

pop → F

push G, 4

pop → C

pop → B

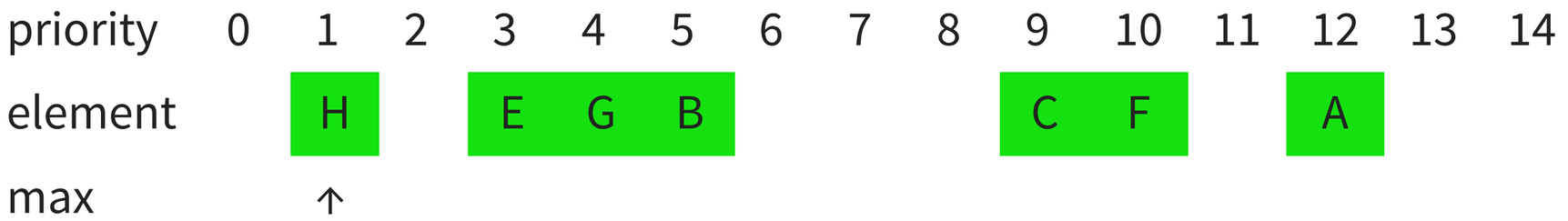
pop → G

push H, 1

pop → E

→ pop

.




```
push A, 12
push B, 5
push C, 9
pop          → A
push E, 3
push F, 10
pop          → F
push G, 4
pop          → C
pop          → B
pop          → G
push H, 1
pop          → E
pop          → H
→ .
```

# Implementation details

- good for branch predictor
- great for caches
- we can store, additionally, an array of $P$ bits (“bitmap”)
  - bit $p$ set to one if there is an element with priority $p$
  - makes “pop” operations essentially 64x faster

# Applications of **bitmap priority queues**

- Linux kernel: scheduling parallel tasks
- Linear algebra: sparse matrices

# Sort operations

# Heap sort

- Push $n$ elements to heap: $O(n \log n)$
- Pop $n$ elements one by one: $O(n \log n)$

$\Rightarrow$ $O(n \log n)$ worst case

# Comparison sort methods

| Method | Average | Worst case | Additional storage | Combines with insert. sort |
|---|---|---|---|---|
| Quicksort | $O(n \log(n))$ | $O(n^2)$ | none | yes |
| Merge sort | $O(n \log(n))$ | $O(n \log(n))$ | $n$ | yes |
| Heap sort | $O(n \log(n))$ | $O(n \log(n))$ | none | no |

# Special case 1

- Assume that
  - we sort $n$ elements with priorities $S \subseteq \{0, \ldots, P-1\}$
  - no two elements have the same priority (hence $P \geq n$)
- Then,
  - we represent the elements as a bitmap priority queue
  - Push: $O(n)$
  - Pop: $O(P)$

$\Rightarrow \quad O(n+P)$

# Special case 2: counting sort

- Assume that
  - we sort $n$ elements with priorities $S \subseteq \{0, \ldots, P-1\}$
  - $P \leq n$ (we may have duplicates)
- Then,
  - we allocate a static array `count` of size $P$
  - we allocate a static array `result` of size $n$
  - we count the number of occurences of each priority: $O(n)$
  - we sweep `count` backwards to determine offsets: $O(P) = O(n)$
  - we construct the sorted `result` list: $O(n)$

$\Rightarrow \quad O(n)$

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| ↑ | | | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| ↑ | | | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | ↑ | | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| | ↑ | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | ↑ | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | ↑ | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | | |
| | | ↑ | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| | ↑ | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | ↑ | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 2 | 0 | 0 | 1 | 0 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| | ↑ | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | ↑ | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 3 | 0 | 0 | 1 | 0 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | ↑ | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 3 | 0 | 0 | 1 | 1 | | |
| | | | | | | | | ↑ | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | ↑ | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 3 | 0 | 1 | 1 | 1 | | |
| | | | | | | ↑ | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | ↑ | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 1 | 0 | 3 | 0 | 1 | 2 | 1 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | ↑ | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 2 | 0 | 3 | 0 | 1 | 2 | 1 | | |
| | | ↑ | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Count

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 2 | 0 | 3 | 0 | 1 | 3 | 1 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 | |
| | | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | |
| count value | 0 | 2 | 0 | 3 | 0 | 1 | 3 | 1 | | | |
| | | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| result | | | | | | | | | | | |
| ↑ | | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 2 | 0 | 3 | 0 | 1 | 4 | 1 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 | |
| | | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | |
| count value | 0 | 2 | 0 | 3 | 0 | 5 | 4 | 1 | | | |
| | | | | | | ↑ | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| result | | | | | | | | | | | |
| ↑ | | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 2 | 0 | 3 | 5 | 5 | 4 | 1 | | |
| | | | | | ↑ | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 | |
| | | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | |
| count value | 0 | 2 | 0 | 8 | 5 | 5 | 4 | 1 | | | |
| | | | | ↑ | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| result | | | | | | | | | | | |
| ↑ | | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 0 | 2 | 8 | 8 | 5 | 5 | 4 | 1 | | |
| | | | ↑ | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 | |
| | | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | |
| count value | 0 | 10 | 8 | 8 | 5 | 5 | 4 | 1 | | | |
| | | ↑ | | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| result | | | | | | | | | | | |
| ↑ | | | | | | | | | | | |

# Determine offsets

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 | |
| | | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | | |
| count value | 10 | 10 | 8 | 8 | 5 | 5 | 4 | 1 | | | |
| | ↑ | | | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| result | | | | | | | | | | | |
| ↑ | | | | | | | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| ↑ | | | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 10 | 8 | 8 | 5 | 5 | 4 | 1 | | |
| ↑ | | | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | | | |
| ↑ | | | | | | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | ↑ | | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 10 | 8 | 7 | 5 | 5 | 4 | 1 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | | | | | 3 | | |
| | | | | | | | | ↑ | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | ↑ | | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 10 | 8 | 7 | 5 | 5 | 3 | 1 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | 6 | | | | 3 | | |
| | | | | ↑ | | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | ↑ | | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 7 | 5 | 5 | 3 | 1 | | |
| | | ↑ | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | 6 | | | | 3 | | 1 |
| | | | | | | | | | | ↑ |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | ↑ | | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 6 | 5 | 5 | 3 | 1 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | 6 | | | 3 | 3 | | 1 |
| | | | | | | | ↑ | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | ↑ | | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 5 | 5 | 5 | 3 | 1 | | |
| | | | | ↑ | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | | | | 6 | | 3 | 3 | 3 | | 1 |
| | | | | | | ↑ | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | ↑ | | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 5 | 5 | 5 | 3 | 0 | | |
| | | | | | | | | ↑ | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | | | 6 | | 3 | 3 | 3 | | 1 |
| | ↑ | | | | | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | ↑ | | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 5 | 5 | 4 | 3 | 0 | | |
| | | | | | | ↑ | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | | | 6 | 5 | 3 | 3 | 3 | | 1 |
| | | | | | ↑ | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | ↑ | | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 9 | 8 | 5 | 5 | 4 | 2 | 0 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | | 6 | 6 | 5 | 3 | 3 | 3 | | 1 |
| | | | ↑ | | | | | | | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | ↑ | |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 8 | 8 | 5 | 5 | 4 | 2 | 0 | | |
| | | ↑ | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | | 6 | 6 | 5 | 3 | 3 | 3 | 1 | 1 |
| | | | | | | | | | ↑ | |

# Construct result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 8 | 8 | 5 | 5 | 4 | 1 | 0 | | |
| | | | | | | | ↑ | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | 6 | 6 | 6 | 5 | 3 | 3 | 3 | 1 | 1 |
| | | ↑ | | | | | | | | |

# Result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| priority | 3 | 6 | 1 | 3 | 3 | 7 | 5 | 6 | 1 | 6 |
| | | | | | | | | | | ↑ |
| count index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| count value | 10 | 8 | 8 | 5 | 5 | 4 | 1 | 0 | | |
| ↑ | | | | | | | | | | |
| result index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| result | 7 | 6 | 6 | 6 | 5 | 3 | 3 | 3 | 1 | 1 |
| ↑ | | | | | | | | | | |