# Intersection Cuts for Single Row Corner Relaxations

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**Abstract**

We consider the problem of generating inequalities that are valid for one-row relaxations of a simplex tableau, with the integrality constraints preserved for one or more non-basic variables. These relaxations are interesting because they can be used to generate cutting planes for general mixed-integer problems. We first consider the case of a single non-basic integer variable. This relaxation is related to a simple knapsack set with two integer variables and two continuous variables. We study its facial structure by rewriting it as a constrained two-row model, and prove that all its facets arise from a finite number of maximal $(\mathbb{Z} \times \mathbb{Z}_+)$-free splits and wedges. The resulting cuts generalize both MIR and 2-step MIR inequalities. Then, we describe an algorithm for enumerating all the maximal $(\mathbb{Z} \times \mathbb{Z}_+)$-free sets corresponding to facet-defining inequalities, and we provide an upper bound on the split rank of those inequalities. Finally, we run computational experiments to compare the strength of wedge cuts against MIR cuts. In our computations, we use the so-called trivial fill-in function to exploit the integrality of more non-basic variables. To that end, we present a practical algorithm for computing the coefficients of this lifting function.

**Keywords** lifting · cutting planes
**Mathematical Subject Classification** 90C11 · 90C57

## 1 Introduction

Since the beginnings of integer programming, cut-generating functions [25] and intersection cuts [6] have provided a theoretical foundation for computing a wide range of valid inequalities. Yet, the most important classes of general-purpose cutting planes used in practice, such as Gomory mixed-integer cuts [23] and mixed-integer rounding inequalities [33, 32], are generated either from a single row, or from a single linear combination of rows of the simplex tableau. However, a 2007 paper from Andersen, Louveaux, Weismantel and Wolsey [4] triggered a renewed interest in the study of inequalities that can only be generated when considering two or

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more tableau rows simultaneously. More specifically, some form of the following mixed-integer model was studied in [24, 6, 8, 4, 15, 12, 18, 10, 22, 11, 31]:

$$\begin{aligned} & x = f + \sum_{j \in N} r^j s_j, \\ & x \in S, \\ & s_j \in \mathbb{R}_+, \text{ for all } j \in N \end{aligned} \tag{1}$$

where $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$, $r^j \in \mathbb{Q}^m$ for $j \in N$, and $S$ is the set of integral points contained in some rational polyhedron in $\mathbb{R}^m$. The usual approach to obtain this model from a general MIP is to consider some simplex tableau of its LP relaxation, then drop the rows in which the basic variable is continuous and relax integrality constraints on non-basic variables. The latter constraints, however, can be exploited by adopting a *lifting* approach [13, 17, 20, 19, 9]: First, the integral non-basic variables are fixed to zero. This amounts to removing the corresponding columns from the problem, yielding a model of the form (1). A facet-defining inequality is generated for this model. Then, the missing variables are re-introduced, and corresponding valid coefficients are computed, while the coefficients of the continuous variables are kept unchanged. In other words, an initial inequality $\alpha^T s \geq 1$ is *lifted* into a higher-dimensional space, yielding an inequality $\gamma^T y + \alpha^T s \geq 1$ that is valid for

$$\begin{aligned} & x = f + \sum_{j \in K} r^j s_j + \sum_{j \in N} r^j s_j, \\ & x \in S, \\ & s_j \in \mathbb{Z}_+, \text{ for all } j \in K, \\ & s_j \in \mathbb{R}_+, \text{ for all } j \in N. \end{aligned} \tag{2}$$

Given $\alpha$, a lifting $\gamma$ is said to be *minimal* if there does not exist a valid inequality ${\gamma'}^T y + \alpha^T s \geq 1$ for (2) that is distinct from $\gamma^T y + \alpha^T s \geq 1$ and dominates it. Furthermore, the lifting is *unique* (or *sequence-independent*) if there does not exist a valid inequality ${\gamma''}^T y + \alpha^T s \geq 1$ for (2) that is distinct from $\gamma^T y + \alpha^T s \geq 1$ and minimal.

Note that with this approach, even in the simplest case where a unique minimal lifting exists and can be computed, not all facet-defining inequalities for (2) can be obtained; only those inequalities for which the $\alpha$ coefficients form a facet-defining inequality for (1). Instead, we are interested in characterizing more facet-defining inequalities of (2); in particular, inequalities that can not be obtained through lifting of facet-defining inequalities for the continuous model (1). In this paper, we focus on the single-row case ($m = 1$) with $S = \mathbb{Z}$.

We start by considering the special case where there is a single integral non-basic variable ($|K| = 1$). We assume for the sake of conciseness that we have continuous variables with both positive and negative coefficients. The model can then be simplified by aggregating them according to the sign of their coefficient. That is, we study the structure of the set

$$P = \left\{ (x, s) \in \mathbb{Z} \times \mathbb{R}^3_+ \ : \ x = \phi + \rho s_1 + s_2 - s_3, \ s_1 \in \mathbb{Z} \right\}.$$

By considering $s_3$ as the slack of an inequality constraint, we can see that $P$ is closely related to the set of solutions of a mixed-integer knapsack problem having two integral variables and one continuous variable. Hirschberg and Wong [29] developed a polynomial-time algorithm to optimize over pure integer knapsack problems with two variables. Agra and Constantino [1, 2] provided a complete characterization of $\mathrm{conv}(P)$, and a polynomial-time method exploiting the approach in [29] to enumerate its facet-defining inequalities. Similar results are also due to Atamtürk and Rajan [5]. The particularity of our approach is that we use the framework of multi-row intersection cuts [6]. By doing so, we obtain a nice geometric interpretation of our results. In particular, this yields a natural upper bound on the split rank of the integer hull of $P$. Moreover, the tools we develop are particularly well-suited for a practical implementation, and we present computational results using our cuts on MIPLIB 2010 [30] instances.

In Section 2, we rewrite $P$ as a two-row model, and show that all the facet-defining inequalities for $\mathrm{conv}(P)$ are intersection cuts obtained either from a split unbounded along the line $\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right) + \lambda \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right)$, or from a finite number of wedges whose vertex lies on the same line. In Section 3, we present an algorithm to enumerate all the sets that yield facet-defining intersection cuts. We exploit this algorithm in Section 4 to compute an upper bound on the split rank of the corresponding facets.

Next, we tackle the problem of exploiting integrality constraints on more non-basic variables. Our approach is more traditional in this case. We compute the cut coefficients of the additional integral variables by making use of the trivial lifting [26, 7] (or trivial fill-in [17, 20]) function. Dey and Wolsey [17, 20] show how this can be done when the trivial lifting is the unique minimal lifting. In general however, this requires solving a mixed-integer programming problem with two integer variables. Albeit of polynomial-time complexity in theory [28], this problem can be expensive to solve with a general-purpose solver, particularly if one thinks that the lifting must be done for every integer variable and for every cut. In Section 5, we present a practical algorithm to compute these coefficients. Finally, in Section 6, we run computational experiments to compare the strength of the cuts developed here against MIR cuts [33] (which they generalize) alone. Our results indicate that for some instances, we close significantly more gap than MIR with our one-row cuts.

## 2 The case of a single integral non-basic variable

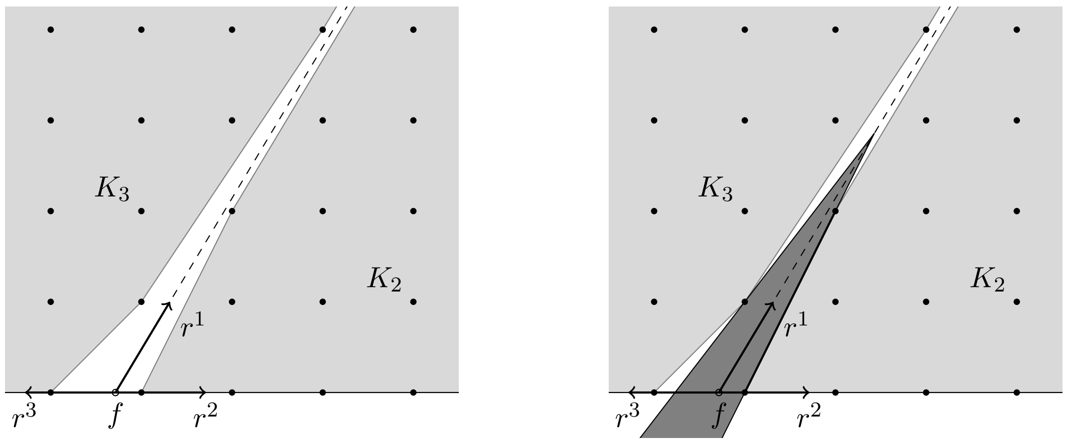
In this section, we start by considering a one-row model where the integrality of a single non-basic variable is preserved. More precisely, we study the structure of the set

$$\left\{(x, s) \in \mathbb{Z} \times \mathbb{R}^3_+ \;:\; x = \phi + \rho s_1 + s_2 - s_3,\ s_1 \in \mathbb{Z}\right\},$$

where $f \in \mathbb{Q} \setminus \mathbb{Z}$ and $\rho \in \mathbb{Q}$. As suggested by Conforti, Cornuéjols and Zambelli [13], this set can be rewritten as

$$P_I = \left\{(x, s) \in S \times \mathbb{R}^3_+ : \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right) s_1 + \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) s_2 + \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) s_3\right\},$$

where we let $S := (\mathbb{Z} \times \mathbb{Z}_+)$. Note that we use $S = (\mathbb{Z} \times \mathbb{Z}_+)$ to emphasize that $x_2$ is nonnegative, although $S = \mathbb{Z}^2$ would yield the same set since $x_2 = s_1$ and $s_1 \geq 0$. We now have a two-row model $P_I$ for which all $s$ variables are continuous. Let $f = \left(\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right)$, $r^1 := \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right)$, $r^2 := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $r^3 := \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$




(a) Knapsack sets $K_2$ and $K_3$. (b) Example of facet-defining wedge.

Figure 1: Knapsack sets and facet-defining $S$-free sets.

and $R := [r^1|r^2|r^3]$, i.e.,

$$P_I = \left\{(x, s) \in S \times \mathbb{R}^3_+ : x = f + Rs\right\},$$

Our definition of $\operatorname{conv}(P_I)$ is a special case of the set with the same name in [4], and the following properties carry over from [4]:

**Proposition 1.** *[4]*

*(i) The dimension of* $\operatorname{conv}(P_I)$ *is three.*

*(ii) The extreme rays of* $\operatorname{conv}(P_I)$ *are* $(\rho, 1, 1, 0, 0)$*,* $(1, 0, 0, 1, 0)$ *and* $(-1, 0, 0, 0, 1)$*.*

Closely related to the structure of $\operatorname{conv}(P_I)$ are the two knapsack sets

$$K_j = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(f + \operatorname{cone}(r^1, r^j)\right)\right) \qquad \text{for } j \in \{2, 3\},$$

illustrated in Figure 1a. For the vertices of $\operatorname{conv}(P_I)$, we can refine the characterization from [4]:

**Proposition 2.** *A point* $(\bar{x}, \bar{s}) \in P_I$ *is a vertex of* $\operatorname{conv}(P_I)$ *if and only if* $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ *for some* $j \in \{2, 3\}$ *and* $\bar{x}$ *is a vertex of* $\operatorname{conv}(K_j)$*.*

*Proof.* ($\Rightarrow$) Assume that $(\bar{x}, \bar{s})$ is a vertex of $\operatorname{conv}(P_I)$. Then, $\bar{x}$ is integer and $\bar{s}$ is a vertex of $P_I \cap \{(x, s) : x = \bar{x}\}$, hence a basic feasible solution to the system $\{s \in \mathbb{R}^3_+ \ : \ Rs = \bar{x} - f\}$. Thus, $\bar{s}$ has at most two nonzero components. Furthermore, since the submatrix $[r^2|r^3]$ is not invertible, either $s_2$ or $s_3$ is nonbasic, hence zero. Therefore, $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ for some $j \in \{2, 3\}$. Since $\bar{x}$ is integer, this implies that $\bar{x} \in K_j$. We next show that $\bar{x}$ is a vertex of $\operatorname{conv}(K_j)$. Suppose it is not. Then, there must exist $x^1, \ldots, x^k \in K_j$ distinct from $\bar{x}$ and $\lambda \in \mathbb{R}^k_+$ such that $\bar{x} = \sum_{i=1}^k \lambda_i x^i$ and $\sum_{i=1}^k \lambda_i = 1$. Let $M = [r^1|r^j]$. Note that since $\operatorname{lin}(r^1) \neq \operatorname{lin}(r^j)$, $M$ is invertible. For each $i \in \{1, \ldots, k\}$, let $s^i \in \mathbb{R}^3_+$ be such that $s^i = s^i_1 e_1 + s^i_j e_j$ and

$$\begin{pmatrix} s^i_1 \\ s^i_j \end{pmatrix} = M^{-1}(x^i - f).$$

For every $i \in \{1, \ldots, k\}$, $s_1^i, s_j^i \geq 0$ because $x^i \in K_j$, so $(x^i, s^i) \in P_I$. Furthermore, by linearity, $\bar{s} = \sum_{i=1}^k \lambda_i s^i$, thus $(\bar{x}, \bar{s}) = \sum_{i=1}^k \lambda_i (x^i, s^i)$. This contradicts the assumption that $(\bar{x}, \bar{s})$ is a vertex of $\operatorname{conv}(P_I)$.

($\Leftarrow$) Let $(\bar{x}, \bar{s}) \in P_I$ be such that $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ for some $j \in \{2, 3\}$ and $\bar{x}$ is a vertex of $\operatorname{conv}(K_j)$. We prove that $(\bar{x}, \bar{s})$ is a vertex of $\operatorname{conv}(P_I)$. Suppose it is not. Then, there must exist $k$ points $(x^1, s^1), \ldots, (x^k, s^k) \in P_I$ distinct from $(\bar{x}, \bar{s})$ and $\lambda \in \mathbb{R}_+^k$ such that $(\bar{x}, \bar{s}) = \sum_{i=1}^k \lambda_i (x^i, s^i)$ and $\sum_{i=1}^k \lambda_i = 1$. Let $\{h\} := \{2, 3\} \setminus \{j\}$. Since $\bar{s}_h = 0$ and $\lambda \geq 0$, we have $s_h^i = 0$ for all $i$. Therefore $x^i \in K_j$ for all $i$ and these points are all distinct from $\bar{x}$. We can construct $\bar{x}$ as a convex combination of $k$ points $x^1, \ldots, x^k \in K_j$ distinct from $\bar{x}$. This contradicts the assumption that $\bar{x}$ is a vertex of $\operatorname{conv}(K_j)$. □

We now look at the facet-defining inequalities for $\operatorname{conv}(P_I)$.

**Proposition 3.** *[4] The facet-defining inequalities of* $\operatorname{conv}(P_I)$ *take the form*

*(i)* $s_j \geq 0$ *for* $j \in \{1, 2, 3\}$,

*(ii)* $\alpha^T s \geq 1$ *for some* $\alpha \geq 0$.

Note that inequalities of the form (i) in Proposition 3, i.e. $s_j \geq 0$ for some $j \in \{1, 2, 3\}$, are called *trivial*, while those of the form (ii) are called *nontrivial*. For the nontrivial inequalities, we have the following further characterization.

**Proposition 4.** *Every nontrivial facet-defining inequality* $\alpha^T s \geq 1$ *of* $\operatorname{conv}(P_I)$ *satisfies* $\alpha_2 > 0$ *and* $\alpha_3 > 0$. *If* $\alpha_1 = 0$, *then there are no integer points on the ray* $f + \operatorname{cone}(r^1)$, *and there is only one facet-defining inequality of that form.*

*Proof.* Let $z^2 := (\lceil \phi \rceil, 0, 0, \lceil \phi \rceil - \phi, 0)$ and $z^3 := (\lfloor \phi \rfloor, 0, 0, 0, \phi - \lfloor \phi \rfloor)$. Since $z^2$ and $z^3$ belong to $P_I$, we must have $\alpha_2 > 0$ and $\alpha_3 > 0$, respectively. Suppose $f + \lambda r^1 = \bar{x} \in \mathbb{Z}^2$ for some $\lambda \in \mathbb{R}_+$. Since $\phi \notin \mathbb{Z}$ we have $\lambda > 0$. Then $(\bar{x}_1, \bar{x}_2, \lambda, 0, 0) \in P_I$, and therefore $\alpha_1 > 0$. It follows that if $\alpha_1 = 0$, then $f + \lambda r^1 = \bar{x} \in \mathbb{Z}^2$ does not exist. Finally, we show uniqueness for a facet-defining inequality with $\alpha_1 = 0$. Suppose that $\alpha_2 s_2 + \alpha_3 s_3 \geq 1$ and $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$ are facet-defining for $\operatorname{conv}(P_I)$. Consider the vertices of $\operatorname{conv}(P_I)$ that are tight on $\alpha_2 s_2 + \alpha_3 s_3 \geq 1$. By Proposition 2, they all have $s_h = 0$ for some $h \in \{2, 3\}$. However, the value of $h$ is not the same for all of them, otherwise we could set $\alpha_h = 0$ and the resulting inequality would cut off $z^h$. Let $(\bar{x}, \bar{s})$ be one such vertex and let $\{j\} := \{1, 2\} \setminus \{h\}$. Since $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$ is valid, $\alpha_j' \geq \alpha_j$. By applying the process to all vertices, then repeating for those that are tight on $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$, we obtain $\alpha_2' = \alpha_2$ and $\alpha_3' = \alpha_3$. □

Our motivation for studying a model of the form of $P_I$ is that such model is an ideal setting for computing and using intersection cuts [6]. Specifically, every nontrivial valid inequality for $P_I$ is an intersection cut from some $S$-free set in $\mathbb{R}^2$ [18]. A convex set $B \subseteq \mathbb{R}^m$ is *$S$-free* if its interior contains $f$ but no point of $S$. The set is *maximal* if it is not properly contained into any other $S$-free set. Maximal sets are the only ones that interest us, since any non-dominated inequality can be obtained from such sets. Note that Basu et al. [10] proved that every maximal $S$-free set is polyhedral, and given a polyhedral $S$-free set $B := \left\{ x \in \mathbb{R}^m \ : \ g_i^T (x - f) \leq 1, \ i = 1, \ldots, k \right\}$, the intersection cut coefficient for $s_j$ is given by $\psi_B(r^j) = \max_{i=1,\ldots,k} g_i^T r^j$ [18]. In the context of $\mathrm{conv}(P_I)$, $x \in S = \mathbb{Z} \times \mathbb{Z}_+$ and $s \in \mathbb{R}^3_+$. Proposition 5 shows that in this case, we may restrict our attention to $S$-free sets $B$ with two faces, i.e. $k = 2$. An analogous result was obtained in [13] for an infinite relaxation of $P_I$.

**Proposition 5.** *If $\alpha^T x \geq 1$ is a nontrivial valid inequality for $P_I$, then there exists an $S$-free set*

$$B = \left\{ x \in \mathbb{R}^2 \ : \ g_1^T (x - f) \leq 1, \ g_2^T (x - f) \leq 1 \right\}$$

*such that $\alpha^T x \geq 1$ is the intersection cut computed from $B$.*

Proposition 5 has a very simple justification: Only the intersections (if any) of the facets of $B$ with the line $\mathrm{lin}(r^j)$ affect the intersection cut coefficient $\alpha_j$. Therefore, for a given cut $\alpha \in \mathbb{R}^3_+$, and one can easily construct a wedge or a split in $\mathbb{R}^2$ that provides the three desired intersections. It implies that all facet-defining inequalities for $\mathrm{conv}(P_I)$ can be obtained from maximal $S$-free splits unbounded along the line $f + \mathrm{lin}(r^1)$ and maximal $S$-free wedges with vertex on that same line. As this reasoning relies on a geometric intuition for intersection cuts, we also provide a formal proof.

*Proof of Proposition 5.* The proof is constructive. Let $\alpha^T x \geq 1$ be a nontrivial valid inequality for $P_I$. By Proposition 3, $\alpha \geq 0$, and by Proposition 4, $\alpha_2, \alpha_3 > 0$. We let $g_1 := (\alpha_2, \ \alpha_1 - \rho\alpha_2)$ and $g_2 := (-\alpha_3, \ \alpha_1 + \rho\alpha_3)$. It is straightforward to verify that $B$ then yields the appropriate intersection cut coefficients. Suppose that $B$ is not $S$-free. Then, there exists $\bar{x} \in S$ such that $g_1^T (x - f) < 1$ and $g_2^T (x - f) < 1$. We construct $\bar{s}$ such that $(\bar{x}, \bar{s}) \in P_I$. By substituting $x - f = Rs$ in the two above inequalities, we obtain $\alpha_1 \bar{s}_1 + \bar{s}_2\alpha_2 - \bar{s}_3\alpha_2 < 1$ and $\alpha_1 \bar{s}_1 - \bar{s}_2\alpha_3 + \bar{s}_3\alpha_3 < 1$, respectively. We can assume without loss of generality that either $\bar{s}_2 = 0$ or $\bar{s}_3 = 0$. In each case, one of the latter inequalities yields $\alpha^T \bar{s} < 1$, which contradicts the validity of $\alpha^T x \geq 1$ for $P_I$. $\square$

An interesting feature of the set $B$ constructed above is that a vertex $(\bar{x}, \bar{s})$ of $P_I$ is tight on $\alpha^T s \geq 1$ if and only if $\bar{x}$ is on the boundary of $B$. Indeed, the latter implies either $g_1^T (x - f) = 1$ (if $\bar{s}_3 = 0$), or $g_2^T (x - f) = 1$ (if $\bar{s}_2 = 0$). Again, substituting $x - f = Rs$ yields $\alpha^T \bar{s} = 1$ in both cases.

We now prove that we can restrict our attention even further, to a specific finite family of splits and wedges. This will let us develop an algorithm to enumerate all these relevant $S$-free sets in Section 3. Proposition 4 states that if $\alpha^T s \geq 1$ is facet-defining for $\mathrm{conv}(P_I)$, then $\alpha_2, \alpha_3 > 0$. If $\alpha_1 = 0$, then there is exactly one facet-defining inequality of that form. The proof of Proposition 5 gives us the split set $B = \{x \in \mathbb{R}^2 : \frac{1}{\alpha_2} \leq \binom{-1}{\rho} (x - f) \leq \frac{1}{\alpha_3}\}$[1]. Otherwise, $\alpha > 0$ and $B$ is a wedge with its apex on the line $f + \mathrm{lin}(r^1)$. Then, Theorem 6 gives a useful characterization of the corresponding facet-defining inequalities.

**Theorem 6.** *(i) A valid inequality $\alpha^T s \geq 1$ where $\alpha > 0$ is facet-defining for* $\mathrm{conv}(P_I)$ *if and only if it is tight at three distinct vertices of* $\mathrm{conv}(P_I)$*. (ii) Furthermore, at least one of those three vertices corresponds to a vertex of* $\mathrm{conv}(K_2)$*, and at least one corresponds to a vertex of* $\mathrm{conv}(K_3)$.

[1] If $f$ and $\rho$ are rational numbers, we can compute geometrically a maximal lattice-free set of that form. Specifically, letting $d \in \mathbb{Z}$ such that $fd \in \mathbb{Z}$ and $\rho d \in \mathbb{Z}$, $g = \gcd(d, \rho d)$ and $v = \frac{fd}{g} - \left\lfloor \frac{fd}{g} \right\rfloor$, we get the cut $\frac{g}{d(1-v)} s_2 + \frac{g}{dv} s_3 \geq 1$, provided that $\frac{fd}{g} \notin \mathbb{Z}$.

*Proof.* Let $P_s := \operatorname{proj}_s \operatorname{conv}(P_I)$ be the projection of $\operatorname{conv}(P_I)$ on the space of the $s$ variables. (i) $\Leftarrow$: Since $\dim(P_s) = 3$, a valid inequality that is tight at three affinely independent points is facet-defining. (i) $\Rightarrow$: Since $\dim(P_s) = 3$, a facet of $P_s$ may contain fewer than three vertices of $P_s$ only if its affine hull contains an extreme ray of $P_s$. Assume that $\alpha^T s \geq 1$ is a corresponding facet-defining inequality that is tight at $\bar{s} \in P_s$, i.e. $\alpha^T \bar{s} = 1$. Then, $\alpha^T(\bar{s} + e_j) = 1$ for some $j \in \{1, 2, 3\}$, implying that $\alpha_j = 0$. This contradicts $\alpha > 0$. (ii): Assume that three tight vertices $(x^1, s^1), (x^2, s^2), (x^3, s^3)$ of $\operatorname{conv}(P_I)$ correspond to three vertices $x^1, x^2, x^3$ of $\operatorname{conv}(K_j)$, for a single fixed $j \in \{2, 3\}$. Let $\{h\} = \{2, 3\} \setminus \{j\}$. Then, $s^1_h = s^2_h = s^3_h = 0$. The facet-defining inequality of $\operatorname{conv}(P_I)$ that is tight at these three vertices is $s_h \geq 0$ (Proposition 3), contradicting $\alpha > 0$. $\square$

Theorem 6 means that in order to obtain facet-defining intersection cuts for $P_I$, one should focus on $S$-free sets that have at least three $S$ points on their boundary: at least one of each of $K_2$ and $K_3$. This means that each of those $S$-free sets is tight at two points of either $K_2$ or $K_3$. In other words, one of its facets coincides with a facet of either $\operatorname{conv}(K_2)$ or $\operatorname{conv}(K_3)$. See Figure 1b. An analogous result is well-known in the case of an infinite relaxation of $P_I$ [18, 13].

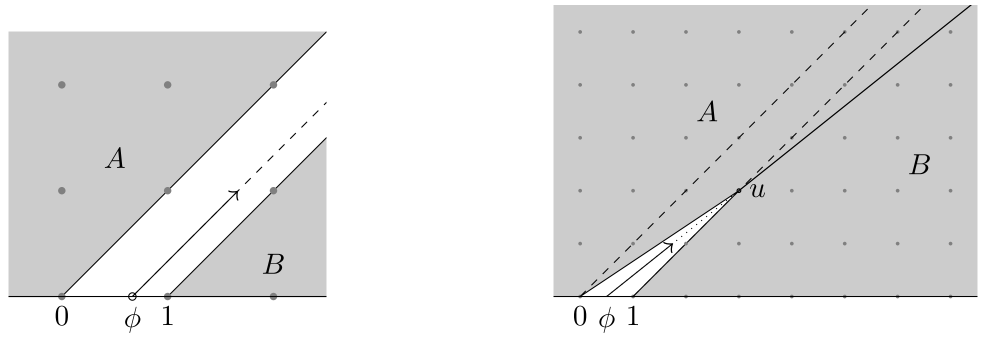
# 3 Enumerating the vertices of the knapsacks

In this section we describe a simple algorithm for enumerating the vertices of the two knapsack sets $K_2$ and $K_3$ described in Section 2, allowing us to enumerate all the splits and wedges that induce facets of $\operatorname{conv}(P_I)$.

Since we have a complete description of the extreme points and rays of $\operatorname{conv}(P_I)$, its facet-defining inequalities could be obtained by enumerating the vertices of its polar, as shown by Andersen, Louveaux, Weismantel and Wolsey [4, 3] in dimension two, and Basu, Hildebrand and Köppe [11] in general dimensions. Although this approach has been performed [31], it has two drawbacks: Even separation in two dimensions relies on optimizing over a cut-generating linear program (CGLP) with the simplex method, which adds a source of numerical inaccuracies. Then, finding *all* facet-defining inequalities would require enumerating the vertices of this CGLP, a difficult computational task. Here, instead, we exploit the characterization provided by Theorem 6 to enumerate the facet-defining inequalities of $\operatorname{conv}(P_I)$.

Enumerating the vertices of the knapsack sets $K_2$ and $K_3$ is a particular case of the integer hull problem. Harvey [28] devised an algorithm for enumerating the facets of the integer hull of an arbitrary two-dimensional polyhedron. The complexity of the algorithm is $O(n \log A_{\max})$ where $n$ is the number of input inequalities and $A_{\max}$ is the magnitude of the largest input coefficient. This algorithm is optimal in the sense that no better asymptotic bound is possible for the problem. In the more specific case of a two-dimensional knapsack set, Agra and Constantino [2, 1] and Atamtürk and Rajan [5] independently gave polynomial-time algorithms. Both are based on the two-dimensional knapsack optimization algorithm of Hirschberg and Wong [29].

Despite the abundant earlier work on the topic, we develop a different method for computing the vertices of the integer hull of a knapsack, with the following motivation. First, our method has a simple geometric interpretation that allows us to prove an upper bound on the split rank of $\operatorname{conv}(P_I)$ (Section 4). Secondly, it is easy to implement and yields a very fast code, which




(a) $\phi = \widehat{\phi + \rho}$ (b) $\phi \neq \widehat{\phi + \rho}$ and $u \in \mathbb{Z}^2$ ($\mathcal{S}$ shown in dashed lines)

Figure 2: Illustration of Propositions 7 and 8.

we use in our computations (Section 6).

Consider the two sets

$$\begin{aligned} A &= \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right) + \operatorname{cone}\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\}\right)\right), \\ B &= \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right) + \operatorname{cone}\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}\right)\right). \end{aligned}$$

Observe that $A$ is simply $\operatorname{conv}(K_3)$ and $B$ is $\operatorname{conv}(K_2)$. We want to obtain the set of vertices of $A$ and of $B$. For simplicity, we assume $0 < \phi < 1$ (if that is not the case, $A$ and $B$ can be translated along the $x_1$ axis to enforce the assumption; the resulting vertices can then be translated back to obtain those of the original sets). An alternative definition of $A$ and $B$ is the following:

$$\begin{aligned} A &= \operatorname{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \leq \phi, x_2 \geq 0\right\}, \\ B &= \operatorname{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \geq \phi, x_2 \geq 0\right\}. \end{aligned}$$

Note that $(0, 0)$ and $(1, 0)$ are always vertices of $A$ and $B$, respectively. In some cases, these are the only vertices of these two sets (Figure 2a).

**Proposition 7.** *If $\phi = \widehat{\phi + \rho}$, then* $\operatorname{vert}(A) = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ *and* $\operatorname{vert}(B) = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$.

*Proof.* First, note that the condition on $\phi$ implies that $\rho \in \mathbb{Z}$. We can, therefore, round down the right-hand side of one of the inequalities that define $A$, to obtain

$$A = \operatorname{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \leq 0, x_2 \geq 0\right\}.$$

Clearly, (0,0) is the only vertex of the linear relaxation of this set. Since the vertex is integral, then the linear relaxation coincides with its integer hull. We conclude that $(0, 0)$ is the only vertex of $A$. To prove that $(1, 0)$ is the only vertex of $B$, we proceed similarly. $\square$

Now suppose that $\phi \neq \widehat{\phi + \rho}$. Then, we have three possible cases, tackled by Proposition 8, 9 and 10. In each case, we determine the facet of $A$ (or $B$) that contains the known vertex $(0, 0)$ (or $(1, 0)$ in the case of $B$) and one additional vertex. Then, Proposition 11 shows that we can repeatedly apply Propositions 7–10 and obtain one new vertex of $A$, $B$ or both at each iteration. The facet of $A$ or $B$ that we are interested in is a split cut based on a lattice-free split $\mathcal{S}$, which is given in all three cases by

$$\mathcal{S} = \left\{ x \in \mathbb{R}^2 : 0 \leq x_1 - \lfloor \phi + \rho \rfloor x_2 \leq 1 \right\}.$$

Let $u \in \mathbb{R}^2$ be the point where the ray $\left(\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right) + \operatorname{cone}\left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right)$ meets the split. When $u$ is an integral point, the vertices of $A$ and $B$ can be easily determined (see Figure 2b).

**Proposition 8.** *If $u \in \mathbb{Z}^2$ then* $\operatorname{vert}(A) = \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), u\}$ *and* $\operatorname{vert}(B) = \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), u\}$.

*Proof.* First, we prove that $u_2 x_1 - u_1 x_2 \leq 0$ is a valid inequality for $A$. We assume that the ray hits the boundary of the split on the "$B$-side", i.e. on the line $x_1 - \lfloor \phi + \rho \rfloor x_2 = 1$. The other case is analogous. Let $x \in A \cap \mathbb{Z}^2$. Since $x$ is not in the interior of the split, it must satisfy either $x_1 - \lfloor \rho + \phi \rfloor x_2 \leq 0$ or $x_1 - \lfloor \rho + \phi \rfloor x_2 \geq 1$. We prove that, in either case, $u_2 x_1 - u_1 x_2 \leq 0$.

First, suppose $x_1 - \lfloor \phi + \rho \rfloor x_2 \leq 0$. Since $u_2 \geq 0$, we can multiply both sides of this inequality by $u_2$ to obtain $u_2 x_1 - \lfloor \phi + \rho \rfloor u_2 x_2 \leq 0$. Also, since $u$ is on the $B$-side boundary of the split, then $u_1 - \lfloor \phi + \rho \rfloor u_2 = 1$. Therefore, $-(u_1 - \lfloor \phi + \rho \rfloor u_2) x_2 \leq 0$. Summing the two previous inequalities, we obtain $u_2 x_1 - u_1 x_2 \leq 0$, as desired.

Now suppose $x_1 - \lfloor \phi + \rho \rfloor x_2 \geq 1$. Since $u$ satisfies $u_1 - \rho u_2 = \phi$ and $u_1 - \lfloor \phi + \rho \rfloor u_2 = 1$, then we must have $u_1 = \frac{\rho - \phi \lfloor \phi + \rho \rfloor}{\rho - \lfloor \phi + \rho \rfloor}, u_2 = \frac{1 - \phi}{\rho - \lfloor \phi + \rho \rfloor}$. Let $\lambda_1 = \frac{1}{\rho - \lfloor \phi + \rho \rfloor}$ and $\lambda_2 = \frac{\phi}{\rho - \lfloor \phi + \rho \rfloor}$. Since $u$ is on the $B$-side boundary of the split, we have $\phi < \widehat{\phi + \rho}$, which implies $\lambda_1, \lambda_2 \geq 0$. Using the previous characterization of $u$, it is straightforward to verify that, if we multiply the valid inequality $-x_1 + \lfloor \phi + \rho \rfloor x_2 \leq -1$ by $\lambda_1$, multiply the valid inequality $x_1 - \rho x_2 \leq \phi$ by $\lambda_2$, and then sum the resulting inequalities, we obtain $u_2 x_1 + u_1 x_2 \leq 0$, as desired.

Since $u_2 x_1 - u_1 x_2 \leq 0$ is valid, we may write

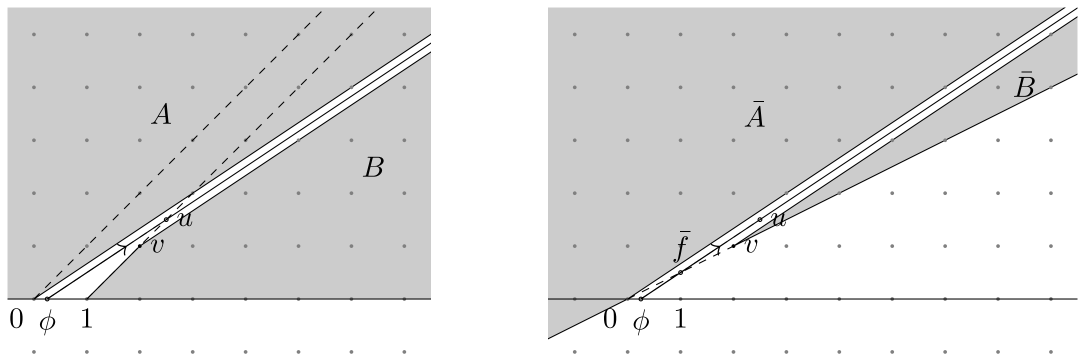
$$A = \operatorname{conv} \left\{ x \in \mathbb{Z}^2 : \begin{array}{l} u_2 x_1 - u_1 x_2 \leq 0 \\ x_1 - \rho x_2 \leq \phi \\ x_2 \geq 0 \end{array} \right\}.$$

It is not hard to see that $(0,0)$ and $u$ are the only vertices of the linear relaxation of this set. Since the linear relaxation has integer vertices, it coincides with its integer hull. We conclude that $(0,0)$ and $u$ are the only vertices of $A$. The proof for $\operatorname{vert}(B)$ is similar. $\square$

Now we consider two more interesting cases, when $u \notin \mathbb{Z}^2$. In the first case, illustrated in Figure 3, the ray hits the boundary of the split on the "$B$-side", i.e. on the line $x_1 - \lfloor \phi + \rho \rfloor x_2 = 1$. We have the following proposition.

**Proposition 9.** *Suppose $u \notin \mathbb{Z}^2$ and $\phi < \widehat{\phi + \rho}$. Let $v$ be the lattice point closest to $u$ in the segment between $u$ and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Let $\bar{f}$ be the intersection between the segment connecting $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ to $v$, and the segment connecting $\left(\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right)$ and $u$. Define*

$$\bar{A} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) - \bar{f}\right\}\right)\right)$$
$$\bar{B} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right), v - \bar{f}\right\}\right)\right)$$




(a) $A$, $B$, $u$, $v$ and $\mathcal{S}$ (dashed lines) (b) $\bar{A}$ and $\bar{B}$

Figure 3: Illustration of proposition 9.

*Then* $\operatorname{vert}(A) = \operatorname{vert}(\bar{A})$ *and* $\operatorname{vert}(B) = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\} \cup \operatorname{vert}(\bar{B})$.

*Proof. (i)* We prove that $x \notin \operatorname{vert}(A)$ if and only if $x \notin \operatorname{vert}(\bar{A})$, for every $x \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ such that $x \notin \operatorname{vert}(A)$. First, suppose $x \in A$. Then there exist $y^1, \ldots, y^k \in A \cap \mathbb{Z}^2 \setminus \{x\}$ such that $x \in \operatorname{conv}\left\{y^1, \ldots, y^k\right\}$. Note, however, that $y^1, \ldots, y^k \in \bar{A}$, since $A \setminus \bar{A}$ is contained in the interior of the split $S$. We conclude that $x$ is not a vertex of $\bar{A}$. Now suppose $x \notin A$. If $x \notin \bar{A}$, then clearly $x \notin \operatorname{vert}(\bar{A})$. We assume, therefore, $x \in \bar{A}$. This implies $x_2 < 0$. Let $y^1 = x + \frac{x_2}{v_2} v, y^2 = x - \frac{x^2}{v_2} v$. Since $v_2 > 0$ and since $v \in \operatorname{rec}(\bar{A})$, it is not hard to see that $y^1, y^2 \in \bar{A}$. Since $x = \frac{1}{2} y^1 + \frac{1}{2} y^2$, we conclude, also in this case, that $x \notin \operatorname{vert}(\bar{A})$.

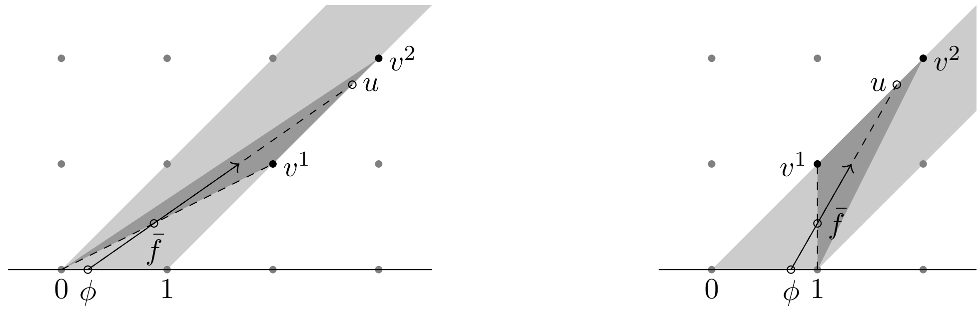
Now let $x \in \mathbb{R}^2$ such that $x \notin \operatorname{vert}(\bar{A})$. If $x \notin A$, then clearly $x \notin \operatorname{vert}(A)$. We assume, therefore, $x \in A$. Also, if $x \notin \bar{A}$, then $x$ belongs to the interior of $S$, hence it is not integer and cannot be a vertex of $A$. Therefore, we also assume that $x \in \bar{A}$. In this case, there exist $y^1, \ldots, y^k \in A \setminus \{x\}$ and $z^1, \ldots, z^l \in \bar{A} \setminus A$ such that $x \in \operatorname{conv}\{y^1, \ldots, y^k, z^1, \ldots, z^l\}$. Now let $\bar{z}^j \in \operatorname{conv}\{x, z^j\}$ such that $\bar{z}^j_2 = 0$, for all $j \in \{1, \ldots, l\}$. It is not hard to prove that $x \in \operatorname{conv}\{y^1, \ldots, y^k, \bar{z}^1, \ldots, \bar{z}^l\}$, and that $y^1, \ldots, y^k, \bar{z}^1, \ldots, \bar{z}^l \in A \setminus \{x\}$. We conclude that $x \notin \operatorname{vert}(A)$.

*(ii)* We prove that, for all $x \in \mathbb{R}^2$, $x \notin \operatorname{vert}(B)$ if and only if $x \neq \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $x \notin \operatorname{vert}(\bar{B})$. First, let $x \in \mathbb{R}^2$ such that $x \notin \operatorname{vert}(B)$. It is easy to see that $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \operatorname{vert}(B)$. Therefore, $x \neq \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Clearly, if $x \notin \bar{B}$ then $x \notin \operatorname{vert}(\bar{B})$, so we assume $x \in \bar{B}$. Then $x \in B$, since $\bar{B} \subseteq B$. Therefore, there exist $y^1, \ldots, y^k \in \bar{B} \setminus \{x\}$ and $z^1, \ldots, z^l \in B \setminus \bar{B}$ such that $x \in \operatorname{conv}\{y^1, \ldots, y^l, z^1, \ldots, z^l\}$. Let $\bar{z}^j \in \operatorname{conv}\{x, z^j\}$ such that $v_2 \bar{z}^j_2 - v_1 \bar{z}^j_1 = 0$. It is not hard to prove that $\bar{z}^1, \ldots, \bar{z}^l \in \bar{B} \setminus \{x\}$, and that $x \in \operatorname{conv}\{y^1, \ldots, y^k, \bar{z}^1, \ldots, \bar{z}^l\}$. We conclude that $x \notin \operatorname{vert}(\bar{B})$.

Now let $x \in \mathbb{R}^2$ such that $x \neq \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $x \notin \operatorname{vert}(\bar{B})$. Clearly, if $x \notin B$ then $x \notin \operatorname{vert}(B)$, so we assume $x \in B$. Furthermore, if $x \in \bar{B}$ then there exist $y^1, \ldots, y^k \in \bar{B} \setminus \{x\}$ such that $x \in \operatorname{conv}\left\{y^1, \ldots, y^k\right\}$. But $y^1, \ldots, y^k \in B$, since $\bar{B} \subseteq B$, hence $x \notin \operatorname{vert}(B)$. Therefore, we also assume $x \notin B$. If $x_2 = 0$, then $x \in \operatorname{conv}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}\lceil x_1 \rceil + 1\\0\end{smallmatrix}\right)\right\}$, hence $x \notin \operatorname{vert}(B)$. If $x_2 > 0$, let

$$y^1 = x + \epsilon \left[v - \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right], y^2 = x - \epsilon \left[v - \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right],$$

where $\epsilon > 0$. It is not hard to prove that, for a small enough $\epsilon$, we have $y^1, y^2 \in B$. Since $x = \frac{1}{2} y^1 + \frac{1}{2} y^2$, we conclude that, in any case, $x \notin \operatorname{vert}(B)$. $\square$




(a) $\phi < \widehat{\phi + \rho}$

(b) $\phi > \widehat{\phi + \rho}$

Figure 4: How $\bar{f}$, $u$, $v^1$ and $v^2$ are found in Proposition 11.

In the second case, the ray hits the boundary of the split on the "$A$-side", i.e. on the line $0 = x_1 - \lfloor \phi + \rho \rfloor x_2$. Then, we have Proposition 10, whose proof we skip since it is analogous to that of Proposition 9.

**Proposition 10.** *Suppose $u \notin \mathbb{Z}^2$ and $\phi > \widehat{\phi + \rho}$. Let $v$ be the lattice point closest to $u$ in the segment between $u$ and $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$. Let $\bar{f}$ be the intersection between the segment connecting $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to $v$, and the segment connecting $\begin{pmatrix}\phi\\0\end{pmatrix}$ and $u$. Define*

$$\bar{A} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), v - \bar{f}\right\}\right)\right)$$
$$\bar{B} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) - \bar{f}\right\}\right)\right)$$

*Then* $\operatorname{vert}(A) = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\} \cup \operatorname{vert}(\bar{A})$ *and* $\operatorname{vert}(B) = \operatorname{vert}(\bar{B})$*.*

The next proposition shows that the vertices of $\operatorname{vert}(\bar{A})$ and $\operatorname{vert}(\bar{B})$ can be computed recursively, after applying an appropriate affine integral unimodular transformation to the coordinate system and scaling of the rays. This leads to a recursive algorithm to compute the vertices of $A$ and $B$. The complete description of a non-recursive version of this algorithm can be found in Appendix A.

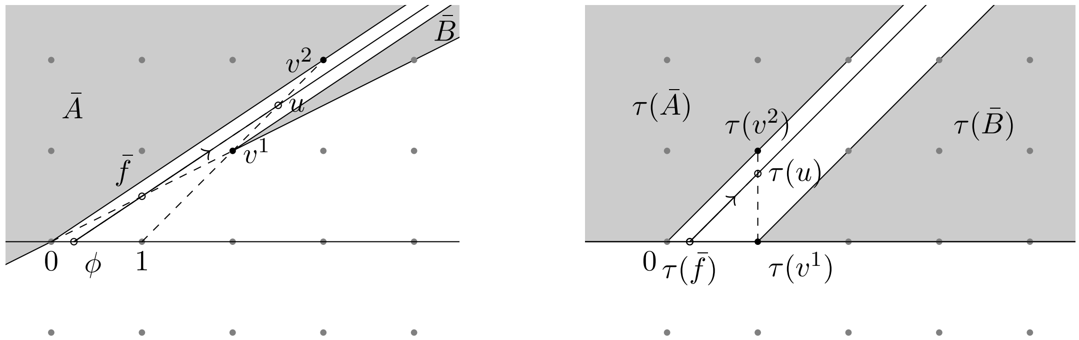
**Proposition 11.** *Suppose $u \notin \mathbb{Z}^2$. If $\phi < \widehat{\phi + \rho}$, let $\bar{A}$ and $\bar{B}$ be defined as in proposition 9. If $\phi > \widehat{\phi + \rho}$, let $\bar{A}$ and $\bar{B}$ be defined as in proposition 10. In either case, there exist $\bar{\phi}, \bar{\rho} \in \mathbb{R}$ and an affine integral unimodular transformation $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\tau(\bar{A}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\begin{pmatrix}\bar{\phi}\\0\end{pmatrix} + \operatorname{cone}\{\left(\begin{smallmatrix}\bar{\rho}\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\}\right)\right),$$
$$\tau(\bar{B}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\begin{pmatrix}\bar{\phi}\\0\end{pmatrix} + \operatorname{cone}\{\left(\begin{smallmatrix}\bar{\rho}\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}\right)\right).$$

*Proof.* Suppose $\phi < \widehat{\phi + \rho}$.

Let $\bar{f}$ and $v \in \mathbb{Z}^2$ as defined in Proposition 9. Let $v^1 = v$ and let $v^2$ be the lattice point closest to $u$ in the half-line $u + \lambda(u - \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), \lambda \geq 0$. That is, $v^1$ and $v^2$ are the closest lattice points to $u$ in the line passing through $u$ and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (see figure 4a).

Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine function such that $\tau\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \tau(v^1) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \tau(v^2) = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. Such a transformation exists, since $v^1$ and $v^2$ are linearly independent. Furthermore, it is integral




(a) Figure 3b zoomed in.

(b) Transformed by $\tau$

Figure 5: Transformation $\tau$ of Proposition 11.

and unimodular, since the triangle defined by $\binom{0}{0}, v^1$ and $v^2$ has integral vertices and its area equals $\frac{1}{2}$. Therefore,

$$\tau(\bar{A}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\tau(\bar{f}) + \operatorname{cone}\left\{\tau\binom{\rho}{1}, \tau\left(\binom{0}{0} - \bar{f}\right)\right\}\right)\right)$$
$$\tau(\bar{B}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\tau(\bar{f}) + \operatorname{cone}\left\{\tau\binom{\rho}{1}, \tau(v^1 - \bar{f})\right\}\right)\right)$$

Since $\bar{f} \in \operatorname{conv}\{\binom{0}{0}, v^1\}$, then $\tau(\bar{f}) \in \operatorname{conv}\{\binom{0}{0}, \binom{1}{0}\}$, which implies that there exists $\bar{\phi} \in \mathbb{R}$ such that $\tau(\bar{f}) = \binom{\bar{\phi}}{0}$. Furthermore, it is not hard to see that there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$ such that

$$\lambda_1 \tau\binom{\rho}{1} = \binom{\bar{\rho}}{1}, \lambda_2 \tau\left(\binom{0}{0} - \bar{f}\right) = \binom{-1}{0}, \lambda_3 \tau\left(v^1 - \bar{f}\right) = \binom{1}{0}$$

This concludes the proof for this case (see Figure 5). When $\phi > \widehat{\phi + \rho}$, the proof is similar, constructing $v^1$ and $v^2$ in an analogous way (see figure 4b), but we let $\tau$ be an affine function satisfying $\tau\binom{1}{0} = \binom{1}{0}, \tau(v^1) = \binom{0}{0}, \tau(v^2) = \binom{0}{1}$, instead. □

Now that we have an algorithm for enumerating the vertices of $A$ and $B$, we finish this section by describing how can we use the previous propositions to get a complete list of maximal $S$-free sets that induce facets of $P_I$.

Definition 12 describes the sequence of $S$-free sets that we construct, one per iteration of the vertex enumeration algorithm. Note that the definition is recursive. Given $\phi$ and $\rho$, Propositions 7–10 show how to compute one $S$-free set $W_u$. Then, Proposition 11 provides an affine transformation $\tau$ and a new model, determined by $\bar{\phi}$ and $\bar{\rho}$, which will yield further $S$-free sets. The sequence $\mathcal{W}(\phi, \rho)$ is constructed by concatenating $W_u$ and the subsequent $S$-free sets $\mathcal{W}(\bar{\phi}, \bar{\rho})$ given by the new model, suitably transformed back into the original space.

**Definition 12.** *Let $\mathcal{W}(\phi, \rho) = \langle W_1, \ldots, W_k \rangle$ be a sequence of sets defined as follows:*

*(i) If the conditions of Proposition 7 are satisfied, then $\mathcal{W}(\phi, \rho) = \langle \mathcal{S} \rangle$, where $\mathcal{S}$ is the split defined previously.*

*(ii) If the conditions of Proposition 8 are satisfied, then $W(\phi, \rho) = \langle W_u \rangle$, where*

$$W_u = u + \operatorname{cone}\left\{\binom{0}{0} - u, \binom{1}{0} - u\right\}.$$

*(iii) Suppose that the conditions of either Proposition 9 or Proposition 10 are satisfied. Let $\bar{\phi}, \bar{\rho}, \tau$ be defined as in Proposition 11, and let*

$$\mathcal{W}(\bar{\phi}, \bar{\rho}) = \left\langle \bar{W}_1, \dots, \bar{W}_l \right\rangle.$$

*Then we define*

$$\mathcal{W}(\phi, \rho) = \left\langle W_u, \tau^{-1}(\bar{W}_1), \dots, \tau^{-1}(\bar{W}_l) \right\rangle,$$

*where $W_u$ is defined as in (ii).*

For every $j \in \{1, \dots, k\}$, it is easy to see that $W_j$ is tight at three integral points; either two vertices of $A$ and one vertex of $B$, or two vertices of $B$ and one vertex of $A$. Observe moreover that, for any combination of three vertices not generated in this fashion, one could not construct a corresponding $S$-free wedge: First, note that the two vertices belonging to the same side must be consecutive, otherwise the wedge cannot be $S$-free. Then, given a pair of tight vertices on one side, the $S$-free wedge that is tight at those vertices and a third on the other side is unique. For every pair of consecutive vertices of either $A$ or $B$ there is a wedge $W_j$ that is tight for these vertices. $W_j$ also has a vertex that is tight on the other side. If we replace this third vertex by any other, the other vertex will either be on the boundary or outside of the initial wedge. In the first case, the new wedge would be identical to the initial one, and in the second case, it would not be $S$-free.

The next proposition shows that $W_j$ is also $S$-free. Then Theorem 6 implies that the intersection cut from $W_j$ yields a facet-defining inequality for $\operatorname{conv}(P_I)$. By Proposition 4, we now have a complete H-description of $\operatorname{conv}(P_I)$.

**Proposition 13.** *Every set in $\mathcal{W}(\phi, \rho)$ is maximal and $S$-free.*

*Proof.* We prove the claim by structural induction. If $\mathcal{W}(\phi, \rho) = \langle S \rangle$ or $\mathcal{W}(\phi, \rho) = \langle W_u \rangle$, then the proposition is clearly true. Now suppose $\mathcal{W}(\phi, \rho) = \left\langle W_u, \tau^{-1}(\bar{W}_1), \dots, \tau^{-1}(\bar{W}_l) \right\rangle$, and suppose, by induction, that $\bar{W}_1, \dots, \bar{W}_l$ are maximal $S$-free sets containing $\binom{\bar{\phi}}{0}$ in their interior. Clearly, $W_u$ is maximal $S$-free and contains $\binom{\phi}{0}$. Let $j \in \{1, \dots, k\}$. We prove that the same holds for $\bar{W}_j$. Since $\bar{W}_j$ is $S$-free, then $\tau^{-1}(W_j)$ does not contain any integral points above the line that connects $(0, 0)$ and $v^1$ (in the first case of Proposition 11) or $(1, 0)$ and $v^1$ (in the second case). Furthermore, the region of $\tau^{-1}(\bar{W}_j)$ that lies below the line is entirely contained in $W_u$. Therefore, $\bar{W}_j$ is $S$-free. Since $\bar{W}_j$ is maximal, it is not hard to see that $\tau^{-1}(W_j)$ is also maximal. $\square$

# 4 Upper bound on the split rank

In this section, we prove that the split rank of $\operatorname{conv}(P_I)$ is at most the sum of the number of vertices of $A$ and the number of vertices of $B$. In order to prove our result, we first need Lemma 14. It shows that given two wedges in a specific configuration and their induced intersection cuts, there is a half-plane where any point cut off by one is cut off by the other.

**Lemma 14.** *Let $W_0$ and $W_1$ be two distinct wedges with their apex on $f + \operatorname{cone}(r^1)$, and intersection cut coefficients $\alpha^0, \alpha^1 \in \mathbb{R}^2$, respectively. Assume that $\alpha^0, \alpha^1 > 0$, that $\alpha^1_1 < \alpha^0_1$,*

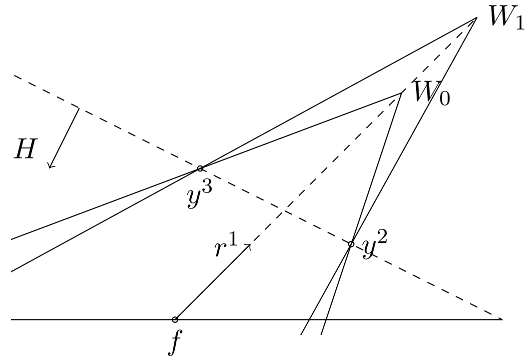



Figure 6: Wedges $W_0$ and $W_1$ in the configuration of Lemma 14.

*and that the faces of the wedges are not pairwise parallel. Let $\{y^2, y^3\}$ be the intersections of their boundaries. Assume also that there exists a closed half-space $H$ that has $\{y^2, y^3\}$ on its boundary and contains neither the apex of $W_0$, nor that of $W_1$ (Figure 6). Then, for any $(\bar{x}, \bar{s}) \in P_I$ such that $\bar{x} \in \text{int}(H)$ and ${\alpha^1}^T \bar{s} < 1$, we also have ${\alpha^0}^T \bar{s} < 1$.*

*Proof.* We first note that given any $\bar{x} \in P_I$, $\bar{x} \in \text{int}(H)$ if and only if ${\alpha^h}^T \bar{s} < 1$ for some $\alpha^h \in \mathbb{R}^3$. Indeed, the rays $(r^1, r^2)$ form a basis of $\mathbb{R}^2$ in which we can observe that $\bar{x} \in \text{int}(H)$ if and only if $\alpha_1^h \bar{s}_1 + \alpha_2^h(\bar{s}_2 - \bar{s}_3) < 1$, for some $\alpha_1^h, \alpha_2^h \in \mathbb{R}$. This is equivalent to $\alpha_1^h \bar{s}_1 + \alpha_2^h \bar{s}_2 + \alpha_3^h \bar{s}_3 < 1$, where $\alpha_3^h = -\alpha_2^h$.

Then, let us consider again a basis $(r^1, r^2)$ of the $x$ space with its origin at $f$. The three lines $\alpha_1^0 s_1 + \alpha_2^0 s_2 = 1$, $\alpha_1^1 s_1 + \alpha_2^1 s_2 = 1$ and $\alpha_1^h s_1 + \alpha_2^h s_2 = 1$ correspond to one face of each of $W_0$ and $W_1$ and $H$, so they intersect in a single point $y^2$. Therefore, $(\alpha_1^0, \alpha_2^0) = \lambda^2(\alpha_1^h, \alpha_2^h) + (1 - \lambda^2)(\alpha_1^1, \alpha_2^1)$, for some $\lambda^2 \in \mathbb{R}$. Similarly, for the other intersection $y^3$, we obtain $(\alpha_1^0, \alpha_3^0) = \lambda^3(\alpha_1^h, \alpha_3^h) + (1 - \lambda^3)(\alpha_1^1, \alpha_3^1)$. for some $\lambda^3 \in \mathbb{R}$. Together, these relationships show $\lambda^2 = \lambda^3$. Let $\lambda := \lambda^2 = \lambda^3$, we get $\alpha^0 = \lambda \alpha^h + (1 - \lambda)\alpha^1$. Since $\alpha_1^1 < \alpha_1^0$ and $H$ does not contain the apex of $W_0$ or $W_1$, we have that $\alpha_1^1 < \alpha_1^0 < \alpha_1^h$, so $\alpha^0$ is not only a linear combination of $\alpha^h$ and $\alpha^1$, but also a convex combination (i.e. $0 \leq \lambda \leq 1$). Therefore, ${\alpha^h}^T s < 1$ and ${\alpha^1}^T s < 1$ together imply ${\alpha^0}^T s < 1$. □

Let $W_1, \ldots, W_k$ be as defined in Subsection 3. For $j \in \{1, \ldots, k\}$, we we define $P_{LP}^j$ as the LP relaxation of $\text{conv}(P_I)$ intersected with the intersection cuts generated from $W_1, \ldots, W_j$. We show that the intersection cut from $W_{j+1}$ has a split rank of one with respect to $P_{LP}^k$.

**Theorem 15.** *For every $j \in \{1, \ldots, k\}$, the intersection cut from $W_j$ has split rank at most $j$.*

*Proof.* We prove the claim by induction. The first wedge $W_1$ has the same intersection points as the split $\mathcal{S}$, so the corresponding cut has split rank 1. Assume now that $W_{j-1}$ yields a cut of split rank $j - 1$ or less. We apply the same reasoning as for the proof of Proposition 13. For any $(x, y)$ such that $y < 0$, we apply Lemma 14 to show that the cut from $W_j$ is implied by the cut from $W_{j-1}$. For any $(x, y)$ such that $y \geq 0$, $W_j$ is included in the split that was considered when generating $W_j$, so it has split rank 1 with respect to $P_{LP}^j$. That LP has split rank at most $j - 1$. So the cut from $W_j$ has split rank at most $j$. □

**Corollary 16.** *Let $k_2$ and $k_3$ be the number of vertices of* $\operatorname{conv}(K_2)$ *and* $\operatorname{conv}(K_3)$*, respectively. The split rank of $P_I$ is at most $k_2 + k_3 - 1$.*

We finish this section by noting that 2-step MIR inequalities [16] can be derived as inequalities for $\operatorname{conv}(P_I)$, and in fact they can be seen as inequalities obtained once the algorithm switches from using Proposition 9 to using Proposition 10 (or vice-versa) for the first time. This suggests that perhaps a lower bound on the split rank may be obtained by considering such cases, since 2-step MIR inequalities have split rank 2. We were, however, unable to derive any such lower bound.

## 5 Multiple Integral Variables via Lifting

We now consider how to obtain valid inequalities for the single-row corner relaxation when the integrality of multiple non-basic variables is preserved. One approach, using the same idea from Section 2, is to study the facial structure of a continuous $(m+1)$-row model, where $m$ is the number of integral non-basic variables. Unfortunately, when considering three or more rows, this relaxation is significantly more complex, and not as well understood. Therefore, we focus instead on *lifting* the valid inequalities we obtained in Section 2.

As mentioned in section 2, the wedges and split sets found in the previous sections are maximal $S$-free sets $B$, which lead to valid inequalities for $\operatorname{conv}(P_I)$ using coefficients $\psi_B$. In order to lift integral variables, let

$$P_I^+ := \left\{ (x, s, z) \in S \times \mathbb{R}^3_+ \times \mathbb{Z}^m_+ : \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \begin{pmatrix} \phi \\ 0 \end{pmatrix} + \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right) s_1 + \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) s_2 + \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) s_3 + \sum_{i=1}^{m} \left(\begin{smallmatrix} \mu_i \\ 0 \end{smallmatrix}\right) z_i \right\},$$

where $m \in \mathbb{Z}_+, \mu \in \mathbb{Q}^m$. Then given a valid inequality

$$\psi \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right) s_1 + \psi \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) s_2 + \psi \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) s_3 \geq 1$$

for $\operatorname{conv}(P_I)$, function $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a *lifting* of $\psi$ if

$$\psi \left(\begin{smallmatrix} \rho \\ 1 \end{smallmatrix}\right) s_1 + \psi \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) s_2 + \psi \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) s_3 + \sum_{i=1}^{m} \pi \left(\begin{smallmatrix} u_i \\ 0 \end{smallmatrix}\right) z_i \geq 1$$

is satisfied by every point in $\operatorname{conv}(P_I^+)$. It is well known that, for $S = \mathbb{R}^2$, if

$$\pi(r) = \min_{k \in \mathbb{Z}^2} \psi(r + k),$$

then $\pi$ is a lifting of $\psi$. This function, introduced by Gomory and Johnson [26], is called the *trivial lifting* of $\psi$. In our case, where $S = \mathbb{Z} \times \mathbb{Z}_+$, it is straightforward to establish that

$$\pi \left(\begin{smallmatrix} \mu \\ 0 \end{smallmatrix}\right) := \min_{k_2 \in \mathbb{Z}_+} \min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu + k_1 \\ k_2 \end{pmatrix} \tag{3}$$

is valid for $\psi$ (Appendix B), although not necessarily minimal (see, e.g. [14] for a definition of a minimal valid function). In the following, we present a finite algorithm that evaluates it for any $\mu \in \mathbb{R}$. During our computational experiments, as discussed in Section 6, we found that this algorithm performs well. It can also be extended to any two-dimensional $S$-free sets.

The following lemma gives us the two main ideas behind the algorithm. First, for any fixed $k_2$, the minimization problem becomes trivial. Secondly, if $k_2$ is fixed at a very large number, then the optimal value also becomes very large. Therefore, these large values of $k_2$ may be safely ignored.

**Lemma 17.** *Let $\bar{k}_2 \in \mathbb{R}$. If $k_1^*$ is an optimal solution to $\min_{k_1 \in \mathbb{R}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix}$, then an optimal solution for*

$$\min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix}$$

*is given by either $\lfloor k_1^* \rfloor$ or $\lceil k_1^* \rceil$. Furthermore,*

$$\min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix} \geq \zeta \bar{k}_2$$

*for some constant $\zeta$.*

*Proof.* The first claim follows from the fact that $\psi$ is a convex function. Furthermore, since $\psi$ is positively homogeneous, we have

$$\min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix} \geq \min_{k_1 \in \mathbb{R}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix} = \bar{k}_2 \left[ \min_{k_1 \in \mathbb{R}} \psi \begin{pmatrix} \frac{\mu+k_1}{\bar{k}_2} \\ 1 \end{pmatrix} \right] = \bar{k}_2 \left[ \min_{\alpha \in \mathbb{R}} \psi \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right].$$

Setting $\zeta := \min_{\alpha \in \mathbb{R}} \psi \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$, we obtain the second claim. ☐

**Algorithm 18** Trivial Lifting

**function** TrivialLifting($\mu$)
    $\eta^* \leftarrow \infty$
    $M \leftarrow \infty$
    $\bar{k}_2 \leftarrow 0$
    $\zeta \leftarrow \min_{\alpha \in \mathbb{R}} \psi \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$
    **while** $\bar{k}_2 < M$ **do**
        $\eta \leftarrow \min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix}$
        **if** $\eta < \eta^*$ **then**
            $\eta^* \leftarrow \eta$
            $M \leftarrow \left\lceil \frac{\eta^*}{\zeta} \right\rceil$
        $\bar{k}_2 \leftarrow \bar{k}_2 + 1$
    **return** $\eta^*$

The full algorithm is described in Algorithm 18. At each iteration of the main loop, we solve the problem

$$\eta := \min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu+k_1 \\ \bar{k}_2 \end{pmatrix}$$

for some fixed value $\bar{k}_2$, starting from zero, and going up to some upper bound $M$, which is initially set to infinity. We also keep track of the smallest optimal value found so far, in the variable $\eta^*$. Every time $\eta^*$ is updated, we also update the upper bound $M$ to $\left\lceil \frac{\eta^*}{\zeta} \right\rceil$. This is justified by Lemma 17, since, for every $\bar{k}_2$ such that $\bar{k}_2 > \left\lceil \frac{\eta^*}{\zeta} \right\rceil$, we have

$$\min_{k_1 \in \mathbb{Z}} \psi \left( \begin{smallmatrix} \mu + k_1 \\ \bar{k}_2 \end{smallmatrix} \right) \geq \zeta \bar{k}_2 \geq \zeta \left\lceil \frac{\eta^*}{\zeta} \right\rceil \geq \eta^*.$$

Therefore, by considering such $\bar{k}_2$, our smallest optimal value $\eta^*$ can never be improved. Note that this algorithm if finite, since, after the very first iteration of the main loop, the upper bound $M$ is no longer infinity.

## 6 Computational Experiments

In order to evaluate the strength of wedge cuts, we implemented a cut generator and tested it on the benchmark set of the MIPLIB 2010. We measured the gap closed by the inclusion of wedge cuts and compared it to the gap closed by considering MIR cuts alone. We also evaluated the speed of the trivial lifting algorithm presented in Section 5.

The cut generator performed the following steps. First, the linear relaxation of the presolved problem was solved, and a certain basic solution with value $z_{LP}$ was obtained. The optimal tableau was stored. Although we solved the relaxation again at a later time, we always used this first optimal tableau to generate all the cuts, hence obtaining only rank-1 cuts. Next, for each row of the tableau corresponding to an integral basic variable, an MIR cut was generated and added to the problem. The strengthened relaxation was then solved again, and another basic solution $x^{MIR}$, with value $z_{MIR}$ was obtained. Then, every possible wedge cut was generated and added to the problem, provided that it cut off the previous solution $x^{MIR}$. More precisely, for each row of the tableau corresponding to an integral basic variable, and for each integral non-basic variable $x_i$ that has non-zero coefficient in that row, we identified the coefficient corresponding to $x_i$ with $\rho$, and generated all the facet-defining wedge and split cuts, as described in Section 2. The cut coefficients for the remaining integral non-basic variables was calculated according to the algorithm from Section 5. Finally, the relaxation was solved again, and a basic solution with value $z_W$ was obtained. In the following, we also denote by $z_{OPT}$ the value of the optimal solution for the original mixed-integer problem.

The cut generator was implemented in C++ and compiled with the GNU C++ Compiler 4.8.4. For the LP solver, we used the library IBM ILOG CPLEX 12.6.2. Considerable care was taken to avoid the generation of invalid cuts. CPLEX was configured for numerical emphasis, and once the LP was solved, each double-precision floating point entry of the resulting tableau was converted to an exact rational number. To avoid the propagation of floating point errors, the enumeration of the facets of the knapsack sets was performed using exact arithmetic, with the help of the GNU Multiple Precision Arithmetic Library 6.1.0 [27]. The cut coefficients were then converted back to double-precision floating point numbers and given to CPLEX. We discarded all cuts with high coefficient dynamism (ratio between the magnitudes of largest and the smallest coefficients of $10^6$ or larger), then considered only the remaining inequalities that cut off the original fractional solution $x^{LP}$ by a significant amount ($10^{-6}$ or more).

Our testbed was the benchmark set of the MIPLIB 2010, which is composed by 87 instances

| Instance | ORIG-GAP (%) | MIR-PERF (%) | W-PERF (%) | W-REL (%) |
|---|---|---|---|---|
| gmu-35-40 | 0.01 | 0.07 | 9.94 | 99.26 |
| eil33-2 | 13.14 | 4.28 | 15.25 | 71.97 |
| neos-1337307 | 0.40 | 3.76 | 6.45 | 41.66 |
| opm2-z7-s2 | 25.29 | 0.62 | 0.98 | 37.17 |
| mik-250-1-100-1 | 19.65 | 53.52 | 73.38 | 27.07 |
| neos-686190 | 23.70 | 4.61 | 5.54 | 16.82 |
| mine-90-10 | 11.15 | 12.40 | 14.51 | 14.60 |
| cov1075 | 14.29 | 3.60 | 4.19 | 13.90 |
| mine-166-5 | 45.09 | 6.57 | 7.58 | 13.35 |
| n3div36 | 12.59 | 16.38 | 18.85 | 13.09 |
| air04 | 1.07 | 8.14 | 9.12 | 10.81 |
| rococoC10-001000 | 34.42 | 21.16 | 22.41 | 5.58 |
| rmine6 | 1.12 | 14.57 | 15.34 | 5.00 |
| reblock67 | 11.61 | 21.38 | 22.46 | 4.81 |
| ran16x16 | 18.48 | 17.25 | 18.07 | 4.50 |
| iis-bupa-cov | 26.40 | 1.22 | 1.26 | 3.59 |
| sp98ir | 1.37 | 4.63 | 4.77 | 2.88 |
| iis-pima-cov | 19.33 | 2.10 | 2.14 | 1.94 |
| iis-100-0-cov | 42.53 | 1.76 | 1.79 | 1.89 |
| eilB101 | 11.64 | 2.64 | 2.69 | 1.82 |
| mzzv11 | 4.86 | 26.99 | 27.11 | 0.43 |
| roll3000 | 13.90 | 21.83 | 21.91 | 0.37 |
| dfn-gwin-UUM | 29.12 | 41.82 | 41.90 | 0.18 |
| csched010 | 18.52 | 3.89 | 3.90 | 0.15 |
| msc98-ip | 1.56 | 17.78 | 17.81 | 0.14 |
| neos-916792 | 17.53 | 4.06 | 4.06 | 0.14 |
| mcsched | 8.56 | 0.04 | 0.04 | 0.08 |
| beasleyC3 | 68.44 | 15.58 | 15.59 | 0.05 |

Table 1: Strength of wedge cuts versus MIR cuts alone.

| Instance | CUTS-MIR | CUTS-W | MIR-T | WEDGE-T | AVG-M |
|---|---|---|---|---|---|
| cov1075 | 582 | 174970 | 0.16 | 0.20 | 13.60 |
| eil33-2 | 30 | 566411 | 7.61 | 8.35 | 32.63 |
| gmu-35-40 | 27 | 58555 | 0.83 | 1.16 | 56.85 |
| mik-250-1-100-1 | 100 | 30221 | 0.17 | 0.28 | 45.73 |
| mine-166-5 | 1436 | 1336080 | 0.29 | 0.57 | 59.54 |
| mine-90-10 | 1875 | 1022638 | 0.18 | 0.38 | 60.88 |
| n3div36 | 48 | 3838798 | 32.06 | 41.67 | 45.83 |
| neos-1337307 | 2263 | 8302981 | 1.13 | 1.52 | 39.20 |
| neos-686190 | 254 | 3162782 | 5.56 | 5.54 | 26.98 |
| opm2-z7-s2 | 7859 | 38797773 | 3.26 | 3.70 | 40.89 |

Table 2: Speed of wedge cuts versus MIR cuts.

of real-world mixed integer programs. For each instance, the following performance indicators were computed:

- ORIG-GAP, the original gap between the first linear relaxation and the original mixed-integer program:
$$\frac{z_{OPT} - z_{LP}}{|z_{OPT}|}$$
- MIR-PERF, the amount of the original gap that was closed by the inclusion of the MIR inequalities:
$$\frac{z_{MIR} - z_{LP}}{z_{OPT} - z_{LP}}$$
- W-PERF, the amount of the original gap that what was closed by the inclusion of all the wedge inequalities:
$$\frac{z_{W} - z_{LP}}{z_{OPT} - z_{LP}}$$
- W-REL, the contribution of the wedge cuts to the gap closure; that is, the amount of the original gap that was closed by wedge inequalities which are not equivalent to MIR inequalities:
$$\frac{z_{W} - z_{MIR}}{z_{W} - z_{LP}}$$

Out of the 87 instances, three were infeasible (`ash608gpia-3col`, `enlight14`, `ns1766074`) and four (`acc-tight5`, `bnatt350`, `m100n500k4r1`, `neos-849702`) had $z_{LP}$ equal to $z_{OPT}$. These instances were not considered. Ten instances exceeded our 60 hour CPU-time limit. Out of the remaining 70 instances, 42 instances presented $z_{MIR} = z_{W}$. Table 1 presents the performance indicators for the remaining 28 instances.

It is well known that, when considering cuts from a single row of the simplex tableau, MIR cuts are very hard to outperform. Indeed, Fukasawa and Goycoolea [21] implemented an exact separator for *knapsack cuts*, a more general set of cuts that includes our wedge cuts, and tested it on the MIPLIB 3.0 and the MIPLIB 2003. Out of the 48 instances processed, on top of MIRs, knapsack cuts increased the gap closure by more than 1 percentage point for only 8 instances, and more than 5 percentage points for only one instance. It should be noted, however, that 44 instances could not be processed due to time constraints in that study.

In our experiment, we obtained noticeably better results. Out of the 70 instances processed, wedge cuts contributed to more than 1% of the gap closure for 20 instances, and more than 5% for 13 instances. In fact, for 5 instances, the contribution from wedge cuts was greater than 25%. For two instances, `gmu-35-40` and `eil33-2`, the percentage was exceptionally high, at 99.26% and 71.97%, respectively. For the instance `gmu-35-40`, MIR cuts alone were only able to close 0.07% of the integrality gap, a negligible amount. The inclusion of wedge cuts improved that closure to 9.94%, which is noticeable. For the instance `mik-250-1-100-1`, although MIR cuts were able to reduce 53.52% of the gap, the inclusion of wedge cuts pushed that reduction to 73.38%, a significant improvement. Therefore, while our results indicate that, for most problems, wedge cuts do not seem to improve the integrality gap significantly when compared to MIR cuts alone, they might be useful for some particular classes of problems.

A side goal of our computational experiment to evaluate the efficiency of the enumeration algorithm presented in Section 3, with the trivial lifting algorithm of Section 5. In order to do that, we run the experiments again for the 10 instances for which wedge cuts presented the best performance, and we collected the additional statistics:

- CUTS-MIR and CUTS-W, the number of MIR cuts and wedge cuts, respectively, generated but not necessarily added to the relaxation,
- MIR-T and WEDGE-T, the average time needed to generate a single MIR cut and a single wedge cut, respectively, in milliseconds,
- AVG-M, the average number of times the inner loop of Algorithm 13 was repeated.

The results are presented on Table 2. On average, the time spent to generate one wedge cut was not much higher than the time spent to generate a single MIR cut. Note, however, that the number of wedge cuts generated, on all instances, was much larger than the number of MIR cuts, since we generate cuts for every tableau row, and for every integral non-basic variable. If wedge cuts are to be used in practice, a better selection of rows and variables is needed.

# 7 Conclusion

In this paper, our main objective was to generate more generic one-row cuts; specifically, cuts that cannot be obtained via the lifting approach. Our strategy was to study cuts that are valid for a relaxation of the simplex tableau with one row and two integer non-basic variables, using the framework of two-row cuts, as suggested by Conforti, Cornuéjols and Zambelli [13]. By doing so, a two-row model with nice properties arises. We developed an algorithm to enumerate all the facet-defining inequalities for this model, which leads to an upper bound on its split rank, and we also developed a practical algorithm for solving the lifting problem that arises when additional integer non-basic variables are present. We implemented all the methods proposed, and performed computational experiments using real-world instances. Our cut generation scheme proved to be very fast in practice. As far as the effectiveness of the cuts is concerned, expectations were limited, since we generate a subset of knapsack cuts, which have been shown by Fukasawa and Goycoolea [21] to be only slightly stronger in practice than the MIR cuts they generalize. During our experiments, for some instances, we obtain a clear improvement in terms of gap closed, over MIR cuts alone.

# A Enumerating the vertices of the knapsacks: complete algorithm

**function** EnumerateVertices($\phi, \rho$)
    $k \leftarrow 0$
    $\phi^0 \leftarrow f$, $\rho^0 \leftarrow \rho$
    $X^A \leftarrow \{(\lfloor \phi \rfloor, 0)\}$, $X^B \leftarrow \{(\lceil \phi \rceil, 0)\}$
    $U^0 \leftarrow I$, $t^0 \leftarrow \mathbf{0}$
    **loop**

**if** $\phi^k == \widehat{\phi^k + \rho^k}$ **then**
  **return** $X^A, X^B$
**else if** $\phi^k < \widehat{\phi^k + \rho^k}$ **then**
  $u_y \leftarrow \frac{1-\phi^k}{\widehat{\phi^k+\rho^k}-\phi^k}$, $u_x \leftarrow 1 + \left\lfloor \phi^k + \rho^k \right\rfloor u_y$
  **if** $u_y \in \mathbb{Z}$ **then**
    **return** $X^A \cup \{U^k u + t^k\}, X^B \cup \{U^k u + t^k\}$
  $v^1 \leftarrow (1 + \left\lfloor \phi^k + \rho^k \right\rfloor \lfloor u_y \rfloor, \lfloor u_y \rfloor)$
  $v^2 \leftarrow (1 + \left\lfloor \phi^k + \rho^k \right\rfloor \lceil u_y \rceil, \lceil u_y \rceil)$
  $X^B \leftarrow X^B \cup \{U^k v^1 + t^k\}$
  Let $W, y$ be a unimodular transformation such that
    $W(0,0) + y = (0,0)$
    $Wv^1 + y = (1,0)$
    $Wv^2 + y = (1,1)$
  $f'_y \leftarrow \frac{\phi^k}{\frac{v^1_x}{v^1_y} - \rho^k}$, $f'_x \leftarrow \frac{v^1_x}{v^1_y} f'_y$
**else if** $\phi^k > \widehat{\phi^k + \rho^k}$ **then**
  $u_y \leftarrow \frac{\phi^k}{\phi^k - \widehat{\phi^k+\rho^k}}$, $u_x \leftarrow \left\lfloor \phi^k + \rho^k \right\rfloor u_y$
  **if** $u_y \in \mathbb{Z}$ **then**
    **return** $X^A \cup \{U^k u + t^k\}, X^B \cup \{U^k u + t^k\}$
  $v^1 \leftarrow (0 + \left\lfloor \phi^k + \rho^k \right\rfloor \lfloor u_y \rfloor, \lfloor u_y \rfloor)$
  $v^2 \leftarrow (0 + \left\lfloor \phi^k + \rho^k \right\rfloor \lceil u_y \rceil, \lceil u_y \rceil)$
  $X^A \leftarrow X^A \cup \{U^k v^1 + t^k\}$
  Let $W, y$ be a unimodular transformation such that
    $Wv^1 + y = (0,0)$
    $W(1,0) + y = (1,0)$
    $Wv^2 + y = (0,1)$
  $f'_y \leftarrow \frac{\phi^k - 1}{\frac{v^1_x - 1}{v^1_y} - \rho^k}$, $f'_x \leftarrow 1 + \frac{v^1_x - 1}{v^1_y} f'_y$
$f^{k+1} \leftarrow Wf' + y$
$\bar{r}^1 \leftarrow Wr^1$
$\rho^{k+1} \leftarrow \bar{r}^1_x / \bar{r}^1_y$
$U^{k+1} \leftarrow U^k W^{-1}$, $t^{k+1} \leftarrow t^k - U^k W^{-1} y$
$k \leftarrow k + 1$
**return** $X^A, X^B$

# B Trivial lifting function for $S$-free sets

In order to simplify the exposition, we examine here the lifting problem in a more general context and adopt the standard approach of the infinite relaxation, as well as its usual notation. We refer the reader to [14] for an introduction.

Let $S := \mathbb{Z}^m \cap Q$, where $Q$ is some rational polyhedron. Let $f \in \mathbb{R}^m \setminus S$. We define

$$R_f := \left\{ y \in \mathbb{R}_+^{\mathbb{R}^m} \ : \ f + \sum_{r \in \mathbb{R}^m} r y_r \in S, \ \ y \text{ has a finite support} \right\},$$

and a lifted version of $R_f$,

$$M_f := \left\{ y \in \mathbb{R}_+^{\mathbb{R}^m}, z \in \mathbb{Z}_+^{\mathbb{R}^m} \ : \ f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} r z_r \in S, \ \ y, z \text{ have a finite support} \right\}.$$

We say that a function $\psi : \mathbb{R}^m \to \mathbb{R}$ is valid for $R_f$ if $\sum_{r \in \mathbb{R}^m} \psi(r) y_r \geq 1$ for all $y \in R_f$. We say that $\psi, \pi : \mathbb{R}^m \to \mathbb{R}$ is valid for $M_f$ if $\sum_{r \in \mathbb{R}^m} \psi(r) y_r + \sum_{r \in \mathbb{R}^m} \pi(r) z_r \geq 1$ for all $(y, z) \in M_f$. Given $\psi$ valid for $R_f$, we say that $\pi$ is a lifting of $\psi$ if $(\psi, \pi)$ is valid for $M_f$. For example, $(\psi, \psi)$ is a lifting of $\psi$.

**Proposition 20.** *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be valid for $R_f$. For any $w : \mathbb{R}^m \to \mathbb{Z}^m \cap \mathrm{rec}(\mathrm{conv}(S))$, the function $\pi(r) := \psi(r + w(r))$ is a lifting of $\psi$.*

*Proof.* For all $(y, z) \in M_f$, we have

$$f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} r z_r \in S.$$

Since $z_r \geq 0$, $w(r) \in \mathbb{Z}^m$, and $w(r) \in \mathrm{rec}(\mathrm{conv}(S))$ for all $r \in \mathbb{R}^m$, we have $x + \sum_{r \in \mathbb{R}^m} w(r) z_r \in S$ for all $x \in S$. In particular,

$$f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} r z_r + \sum_{r \in \mathbb{R}^m} w(r) z_r \in S,$$

i.e.

$$f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} (r + w(r)) z_r \in S.$$

Because $(\psi, \psi)$ is valid for $M_f$, we know that

$$\sum_{r \in \mathbb{R}^m} \psi(r) y_r + \sum_{r \in \mathbb{R}^m} \psi(r + w(r)) z_r = \sum_{r \in \mathbb{R}^m} \psi(r) y_r + \sum_{r \in \mathbb{R}^m} \pi(r) z_r \geq 1,$$

for all $(y, z) \in M_f$. In other words, $(\psi, \pi)$ is valid for $M_f$. $\square$

**Corollary 21.** *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be valid for $R_f$. Then,*

$$\pi(r) := \min_{w \in \mathbb{Z}^m \cap \mathrm{rec}(\mathrm{conv}(S))} \psi(r + w)$$

*is a lifting of $\psi$.*

Proposition 20 only gives sufficient conditions for $\pi$ to be a lifting function. But if we insist on building $\pi$ with a formula of the type $\pi(r) := \psi(r + w(r))$, then in all generality, it is necessary to have $w \in \mathbb{Z}^m \cap \mathrm{rec}(\mathrm{conv}(S))$. Proposition 22 shows that otherwise, we could construct $M_f$ such that $\pi$ is not a lifting.

**Proposition 22.** *Let $S := \mathbb{Z}^m \cap Q$, where $Q$ is some rational polyhedron, and $w \notin \mathbb{Z}^m \cap \operatorname{rec}(\operatorname{conv}(S))$. There exist $f \in \mathbb{R}^m \setminus S$, $d \in \mathbb{R}^m$ and $\psi$ valid for $R_f$ such that if $\pi(d) = \psi(d+w)$, then $\pi$ is not a lifting of $\psi$.*

*Proof.* Since $w \notin \mathbb{Z}^m \cap \operatorname{rec}(\operatorname{conv}(S))$, there exists $\bar{x} \in S$ such that $\bar{x} + w \notin S$. Let $f := \bar{x} + w$. There exists $\varepsilon > 0$ such that $x \notin S$ for all $x \in \mathbb{R}^m$ such that $|x - f| \leq \varepsilon$. Let $\psi(r) := \frac{|r|}{\varepsilon}$. It is easy to verify that $\psi$ is valid for $R_f$. We construct

$$\bar{y}_{-w} := 0, \;\; \bar{z}_{-w} := 1, \;\; \bar{y}_t := 0, \;\; \bar{z}_t := 0, \text{ for all } t \neq -w.$$

Clearly, $f + \sum_{r \in \mathbb{R}^m} r\bar{y}_r + \sum_{r \in \mathbb{R}^m} r\bar{z}_r = \bar{x}$ so $(\bar{y}_r, \bar{x}_r) \in M_f$. However, we can let $d := -w$ and verify that

$$\sum_{r \in \mathbb{R}^m} \psi(r)\bar{y}_r + \sum_{r \in \mathbb{R}^m} \pi(r)\bar{z}_r = \pi(-w) = \psi(0) = 0 \not\geq 1,$$

showing that $(\psi, \pi)$ is not valid for $M_f$. □

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